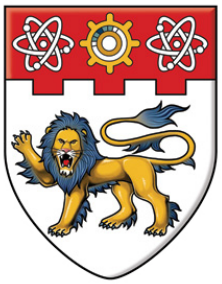

NANYANG TECHNOLOGICAL UNIVERSITY

ENTANGLEMENT DISTRIBUTION

Margherita Zuppardo

School of Physical and Mathematical Sciences

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Abstract

The main focus of this thesis is quantum entanglement and, more specifically, protocols that allow its distribution between distant places.

In the last decades, with the rise of attention on quantum technologies, entanglement is mostly seen as a resource, that could be exchanged, measured, and ideally stored and used when convenient. However, entanglement is not as stable and cheap as the classical resources. One reason for this is that entanglement between two quantum systems is easily disrupted by the interaction with the environment that surrounds them. The presence of noise is even more significant when we wish to entangle two systems that are very far away, a task that is essential for quantum communication technologies. In this case, what can disrupt the entanglement is not only the noise that the particles experience locally, but also the noise in the quantum channel during the communication process, which can be significant over long distances. For this reason, it is important to design distribution protocols that work well in such conditions.

In 2003, Cubitt *et al.* showed the existence of a very curious class of protocols, often called entanglement distribution with separable states (EDSS). They proved that a non-entangled (separable) carrier can be used to increase, or even generate entanglement between two already distant parties. From this, some questions arise. First, what limits or allows an entanglement distribution protocol, if not the entanglement that is sent. Quantum discord was shown to be a necessary, but not sufficient, resource for entanglement distribution. Another natural question is whether these protocols can beat others, for technological purposes, especially in noisy conditions. I will describe some work I have done in an attempt to partially

address both of these questions.

After introducing various tools and concepts that are useful to understand this topic (chapter 1) and motivate the necessity of entanglement distribution in quantum communication (chapter 2), I will describe a proof-of-principle experiment demonstrating EDSS schemes (chapter 3). Later (chapter 4), I will introduce a more general classification of distribution protocols (excessive or non-excessive protocols), based on whether the entanglement gain is larger than the one that is communicated, describing specific examples and general properties of such protocols. I will then partially address the problem regarding the robustness to the noise of communication protocols (chapter 5). Finally, in chapter 6, I will describe how entanglement of a bipartite state can grow with a quantum measurement with unknown results. I discuss the properties of the measurement and of the initial state that make this process effective.

1. Introduction: bipartite entanglement

The goal of this chapter is to introduce a reader with a background in quantum mechanics to some of the main tools and concepts used in the rest of the thesis. For this, I will maintain a quick, often superficial approach. More details can be found in [1, 2] and the rest of the bibliography.

I will discuss the definition and some measures of quantum entanglement of a bipartite system, both in pure and mixed states, and other forms of correlations that two quantum systems can share.

1.1 Pure entangled states

Quantum entanglement is considered one of the strongest, most characterizing, phenomena of quantum mechanics. This characteristic is ubiquitous in any quantum mechanical description, and yet really far from any classical view we are used to.

An entangled pure state of two particles, denoted by X and Y, is defined as a state that cannot be written in the separable form:

$$|\psi\rangle_{XY} = |\phi\rangle_X \otimes |\chi\rangle_Y \,. \tag{1.1.1}$$

One of the most paradigmatic examples is a singlet state of two qubits (quantum systems with 2-dimensional Hilbert space)

$$|\psi_-\rangle = 1/\sqrt{2}\,(|01\rangle - |10\rangle)\,,$$

where $|0\rangle$ and $|1\rangle$ are two orthogonal vectors, forming a basis in the Hilbert space of the each qubit. For instance, if each of the two qubits is a particle with spin 1/2, the state $|0\rangle$ can indicate that the spin along the z axis is $-1/2$, $|1\rangle$ that it is $+1/2$. This state just cannot be written as a single term of the form (1.1.1).

The consequence of having two particles in such an entangled state like the one above is that, whenever a measurement of the spin along z is made on particle X, the value of the spin on particle Y is also known with complete accuracy. However, since the two particles do not interact or communicate with each other at the moment, the state of Y cannot be affected by the measurement on X. Then a measurement on the spin along y can be made at the same time, also with perfect accuracy. In this case, the values of the spin in both directions are known with infinite precision, which would violate the Heisenberg uncertainty principle, since the spin operators in different directions do not commute.

This paradox, or a similar version, was famously noted by Einstein, Podolsky and Rosen in their famous 1935 article "Can Quantum-Mechanical Description of Physical Reality Be Considered Complete?" [3], and is known to most as the EPR paradox. In their paper, they suggested that the description of quantum mechanics, such as the collapse of the wave function or the uncertainty principle, and the notion that the result of a measurement does not exist before the measurement itself, must be somehow incomplete. This opened a debate in which personalities such as Bohr [4] and Schrödinger [5] (who coined the term "entanglement") took part. This debate however mostly ended with the suggestion of John Bell of a way to test the existence of hidden variables in the quantum mechanical description [6], that was eventually convincingly implemented in experiments [7–11].

Entangled states arise very naturally in the dynamics of quantum systems. Often, when two initially separable particles collide (or communicate, or interact), for instance if their interaction Hamiltonian has a form $H_{XY} = h_X \otimes h_Y$, for a finite amount of time, their state in the end will be entangled.

On the other hand, if the Hamiltonian that describes the dynamics of the system is local, i.e. it has the form $H_{XY} = h_Y + h_X$, an initially separable state will always stay separable. The class of operations in which the two subsystems interact

independently that can be made through an Hamiltonian of this form are called "local operations", and it is easy to see that it can be described by a product of unitary operations $U_X \otimes U_Y$, that cannot therefore entangle a separable state.

Entanglement cannot be made by using classical communication either. This means that, even if two operators give each other instructions on how to act on the two particles, for example agree on the time, or other parameters that enter in the local operations, they will never turn an initially separable system into an entangled one. For this reason, it is said that the set of separable states is closed with respect to local operation and classical communication (LOCC).

Another important property of entangled states is that the two subsystems are described by a mixed state. It is known that, for a bipartite state, the correct statistical description of any observable of a single subsystem is given by the matrix obtained by *tracing out* the rest of the system. This means that the state of X is described by the *density matrix*

$$\rho_X = Tr_Y(|\psi\rangle\langle\psi|_{XY}) = \sum_{i=0}^{d_Y-1} \langle\chi_i|\psi\rangle\langle\psi|_{XY}|\chi_i\rangle_Y, \tag{1.1.2}$$

where d_Y is the dimension of the Hilbert space of Y. If the state $|\psi\rangle_{XY}$ is separable, the reduced state of X and Y are also pure, i.e. $\rho_X = |\phi\rangle\langle\phi|_X$, $\rho_Y = |\chi\rangle\langle\chi|_Y$. If the state $|\psi\rangle_{XY}$ is entangled however, the reduced states are mixed i.e. they can be written as

$$\rho_X = \sum_i p_i |\phi_i\rangle\langle\phi_i|_X , \tag{1.1.3}$$

where p_i are positive numbers, or probabilities, such that $\sum_i p_i = 1$.

For example, consider the reduced density matrix of a qubit X that shares the singlet state in eq. (1.1.2) with Y. By tracing out the latter, we get

$$\rho_X = \frac{1}{2}|0\rangle\langle 0| + \frac{1}{2}|1\rangle\langle 1| = \frac{\mathbb{I}}{2}. \tag{1.1.4}$$

This state is completely mixed, meaning that, if one performs any measurement on X, the result is completely undetermined, each eigenvalue has the same probability

1/2 of being measured.

The state of X is completely mixed and, as we will see better later, the amount of mixture in the reduced state is linked to the amount of entanglement in the whole state. The singlet state above is, in fact, one of the maximally entangled states of two qubits.

1.1.1 Schmidt decomposition

In general, a pure state of two subsystems of (finite) dimensions d_X and d_Y can be written as the superposition of $d_X \times d_Y$ states

$$|\psi\rangle_{XY} = \sum_{i=0}^{d_X-1} \sum_{j=0}^{d_Y-1} A_{ij} |\phi_i\rangle_X |\chi_j\rangle_Y. \tag{1.1.5}$$

However, there is particular representation that is often very convenient.

Say that X is a system of lower or equal dimensions than Y (if it's not the case, we can invert the roles of X and Y), $d_X \leq d_Y$. Then, according to the Schmidt theorem [1], there is an orthonormal basis of X, $\{|i\rangle_X\}$ and a set of d_X orthogonal states of Y, $\{|i\rangle_Y\}$, such that the above state can be written as

$$|\psi\rangle_{XY} = \sum_{i=0}^{d_X-1} a_i |i\rangle_X |i\rangle_Y, \tag{1.1.6}$$

where the *Schmidt coefficients* a_i are non-negative, real numbers. The proof is based on the existence of the singular value decomposition of the matrix A_{ij}, and can be found in [1].

As an example, a state of two qubits is usually written as a superposition of four terms:

$$|\psi\rangle_{XY} = A_{00} |00\rangle + A_{01} |01\rangle + A_{10} |10\rangle + A_{11} |11\rangle . \tag{1.1.7}$$

where A_{ij} are in general complex. However, according to the Schmidt theorem, there is a basis of X and a possibly different basis of Y, on which the above state can be

written as the superposition of only two mutually orthogonal states:

$$|\psi\rangle_{XY} = a_0 \left|\hat{0}\bar{0}\right\rangle + a_1 \left|\hat{1}\bar{1}\right\rangle . \tag{1.1.8}$$

For instance, the Schmidt probabilities of the singlet state in equation (1.1.2) are both $1/2$ and the Schmidt basis of X is composed by $|0\rangle$ and $-|1\rangle$. Furthermore, even if Y is not a qubit, but it has arbitrary dimension d_Y, we still only need two terms to describe the state in the Schmidt basis of X.

Here, I list some observations and remarks that are direct consequences of the Schmidt theorem:

- for a given bipartite state, the Schmidt probabilities are unique;
- the real positive numbers $\lambda_i = a_i^2$ sum up to one because of the normalization of the state, and are often called *Schmidt probabilities*;
- a pure bipartite state is separable if and only if its Schmidt decomposition has only one term. Then this is also the separable decomposition;
- the reduced state of X is diagonal on its Schmidt basis: $\rho_X = \sum_{i=0}^{d_X-1} \lambda_i \left|i\right\rangle\!\left\langle i\right|_X$. The reduced state of Y is also diagonal on $|i\rangle_Y$: $\rho_Y = \sum_{i=0}^{d_Y-1} \lambda_i \left|i\right\rangle\!\left\langle i\right|_Y$;
- as a direct consequence of the point above, notice that, for any bipartite pure state, the reduced density matrices ρ_X and ρ_Y have the same spectrum of eigenvalues, given by the Schmidt probabilities λ_i;
- if X and Y share a pure state, the rank of their reduced density matrices is the same, and is at most the smallest between d_X and d_Y, that is the number of terms in the Schmidt decomposition. This is called *Schmidt rank*. Recall that the rank of a matrix is the number of linearly independent rows or columns.

Schmidt probabilities and LOCC operations I already mentioned that separable states cannot be turned into entangled ones by LOCC only. However, if we have two pure bipartite entangled states $|\phi\rangle$ and $|\psi\rangle$, one can sometimes turn one into the other with only LOCC operations.

It was proved [12] that $|\phi\rangle$ can be turned into $|\psi\rangle$ deterministically by LOCC only, or $|\phi\rangle \to |\psi\rangle$, if and only if the vectors of the Schmidt probabilities μ_ψ and λ_ϕ satisfy

$$\lambda_\phi \prec \mu_\psi. \tag{1.1.9}$$

The symbol $\prec$ indicates majorization, which means

$$\sum_{i=0}^{m} \lambda_i \leq \sum_{i=0}^{m} \mu_i \tag{1.1.10}$$

for all m from 0 to the Schmidt rank d_X. Also, in the above equation, it is assumed that the probabilities are arranged in decreasing order, i.e. $\lambda_0 \geq \lambda_1 \geq ... \geq \lambda_{d_X-1}$, and the same for μ_ψ.

Notice that the distribution in which $\mu_0 = 1$, and every other probability is zero majorizes any other distribution, since any sum in equation (1.1.10) cannot exceed 1. This Schmidt decomposition represents a separable state. This implies that not only separable states cannot become entangled via LOCC, but also that, vice versa, any state can become any separable state via LOCC.

On the other hand, the flat distribution $\lambda_i = 1/d_X$ is majorized by any other distribution. This means that the states with flat Schmidt probabilities can be transformed into any other state of the same dimension via LOCC.

All the states that can be obtained with a unitary local transformation from the state

$$|\Psi\rangle_{XY} = \frac{1}{\sqrt{d_X}} \sum_i |i\rangle_X |\phi_i\rangle_Y . \tag{1.1.11}$$

are therefore *maximally entangled states* in the partition $X : Y$, always assuming $d_X \leq d_Y$.

For example, the singlet state is maximally entangled for a space of two qubits, since we saw that its Schmidt probabilities are all equal to 1/2. Any state with 1/2 Schmidt probabilities is maximally entangled in this space. In particular, the four

Bell states:

$$\begin{aligned}
|\phi_+\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right); \\
|\phi_-\rangle &= \frac{1}{\sqrt{2}}\left(|00\rangle - |11\rangle\right); \\
|\psi_+\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle + |10\rangle\right); \\
|\psi_-\rangle &= \frac{1}{\sqrt{2}}\left(|01\rangle - |10\rangle\right).
\end{aligned} \tag{1.1.12}$$

are all maximally entangled and orthogonal to each other. They form a basis of the Hilbert space of two qubits, called *Bell basis*.

If $d_X > 2$, not all the pure states of X and Y can be connected deterministically to each other via LOCC operations. For instance, if $d_X = 4$, consider the two states, in the Schmidt decomposition:

$$\begin{aligned}
|\phi_1\rangle_{XY} &= \sqrt{0.4}\,|00\rangle + \sqrt{0.4}\,|11\rangle + \sqrt{0.1}\,|22\rangle + \sqrt{0.1}\,|33\rangle; \\
|\phi_2\rangle_{XY} &= \sqrt{0.5}\,|00\rangle + \sqrt{0.25}\,|11\rangle + \sqrt{0.25}\,|22\rangle;
\end{aligned} \tag{1.1.13}$$

Although $0.5 > 0.4$, we have $0.5 + 0.25 < 0.4 + 0.4$. Then, neither of these Schmidt probabilities majorizes the other. However, if two qubits in the state

$$|\phi_3\rangle_{AB} = \sqrt{0.4}\,|00\rangle + \sqrt{0.6}\,|11\rangle \tag{1.1.14}$$

are "borrowed", the Schmidt probabilities of the state $|\phi_2\rangle_{XY} \otimes |\phi_3\rangle_{AB}$, in the partition $XA : YB$, are $\{0., 0., 0.1, 0.1, 0.15, 0.15, 0.2, 0.3\}$.
These majorize the probabilities of the state $|\phi_1\rangle_{XY} \otimes |\phi_3\rangle_{AB}$ in the same partition: $\{0.04, 0.04, 0.06, 0.06, 0.16, 0.16, 0.24, 0.24\}$. This means that the state $|\phi_1\rangle_{XY} \otimes |\phi_3\rangle_{AB}$ can be transformed into $|\phi_2\rangle_{XY} \otimes |\phi_3\rangle_{AB}$ with a local unitary operation on XA and YB only. This is called "entanglement catalysis" [13], and it is the phenomenon for which, by "borrowing" an entangled system (in this case A and B in the state $|\phi_3\rangle_{AB}$), a local transformation that was not deterministically possible becomes so. The borrowed system is not affected by the operation, and can therefore be "re-

turned" in the end.

1.1.2 Entropies of a quantum state

Von Neumann entropy

In classical information theory, the Shannon entropy [14]

$$H(X) = -\sum_{i=0}^{d-1} p_i \log_2(p_i) \tag{1.1.15}$$

measures the length of the shortest string of bits needed to encode the message sent by a source associated with a random variable X. Different values of X appear with probabilities p_i. In this sense, it is intuitively clear that such quantity is larger if we consider two correlated variables X and Y instead of just X:

$$H(X,Y) \geq H(X), \tag{1.1.16}$$

because you need a longer string to encode a message coming from two variables instead of one. The equal sign only holds if the variables are perfectly correlated.

The Von Neumann entropy [15, 16] is a direct generalization of the Shannon entropy to a quantum system:

$$S(\rho) = -\mathrm{Tr}\left[\rho \log_2(\rho)\right]. \tag{1.1.17}$$

In fact, if we write ρ in its eigenbasis, we see that, in terms of the eigenvalues of ρ, $\{\lambda_0 ... \lambda_{d-1}\}$, the above expression can be written as $S(\rho) = -\sum_{i=0}^{d-1} \lambda_i \log_2(\lambda_i)$, which is the Shannon entropy of the set of probabilities given by the eigenvalues λ_i.

If $\rho = \rho_X$ is the reduced state of a bipartite pure system, the probabilities λ_i are just the Schmidt probabilities of the pure state. For a pure state, the Von Neumann entropy is $S(|\psi\rangle\langle\psi|_{XY}) = 0$. However, for a subsystem of a pure state, this quantity is in general positive. For instance, consider the pure state of two qubits in the Schmidt decomposition shown in Eq. (1.1.8). The von Neumann entropy of the reduced density matrix of X is $S(\rho_X) = -a_0^2 \log_2(a_0^2) - a_1^2 \log_2(a_1^2)$, that is strictly

positive for any value of the Schmidt coefficient $a_0 \in [0, 1]$. So this state violates the bound imposed to the Shannon entropy of a classical system (1.1.16). Separable states ($a_0 = 0, 1$), on the other hand, respect the bound. This leads us to suppose that the Von Neumann entropy is a good indicator of entanglement in a bipartite pure system. This is in fact true in general, for any finite dimensions of the system. We list some properties of the Von Neumann entropy that reinforce this idea:

- for a bipartite pure state ρ_{XY}, $S(\rho_X) = S(\rho_Y)$;
- $S(\rho) = 0$ if and only if ρ is pure (if $\rho = \rho_X$ describes a subsystem of a bipartite pure state, this means $|\psi\rangle_{XY}$ is separable);
- The maximum value of $S(\rho)$ for a d-dimensional system is $S(\rho) = \log_2(d)$ (if $\rho = \rho_X$ describes the subsystem of a bipartite pure state, this means $|\psi\rangle_{XY}$ is maximally entangled).

Given a state of three subsystems A, B and C, the entropies of the subsystems satisfy the strong sub-additivity condition $S(\rho_{ABC}) + S(\rho_B) \leq S(\rho_{AB}) + S(\rho_{BC})$ [17]. A weaker, but sometimes useful condition derives from that, and from the fact that $S(\rho_{ABC}) \geq 0$. For any state

$$S(\rho_{AC}) \leq S(\rho_A) + S(\rho_C). \tag{1.1.18}$$

This is called sub-additivity condition.

Linear Entropy

The linear entropy [18] of a quantum state is defined as

$$S_L(\rho) = 1 - \mathrm{Tr}(\rho^2). \tag{1.1.19}$$

If the density matrix $\rho = \rho_X$ represents a subsystem of a pure state, this quantity can be expressed in terms of Schmidt probabilities as $S_L(\rho) = 1 - \sum_{i=0}^{d_X - 1} \lambda_i^2$. When applied to a subsystem, this quantity has very similar properties to the Von Neumann entropy:

- for a bipartite pure state ρ_{XY}, $S_L(\rho_X) = S_L(\rho_Y)$;

- $S_L(\rho) = 0$ if and only if ρ is pure, and if the state describes a subsystem, this means that the total state is separable;

- The maximum value of $S_L(\rho)$ for a d-dimensional system is $S(\rho) = 1 - 1/d$. If the state describes a subsystem, this means that the state is maximally entangled;

- Given a state of three subsystems A, B and C, the linear entropies of the subsystems also satisfy the sub-additivity condition [19]

$$S_L(\rho_{AC}) \leq S_L(\rho_A) + S_L(\rho_C). \tag{1.1.20}$$

Example: entanglement of a tripartite system

Let us consider a pure state of three qubits:

$$|\psi\rangle_{ABC} = \frac{1}{\sqrt{2}} |0\rangle_A |\phi_+\rangle_{BC} + \frac{1}{\sqrt{2}} |1\rangle_A |10\rangle_{BC}\,, \tag{1.1.21}$$

where $|\phi_+\rangle$ is the Bell state. This is already explicitly in the Schmidt form for the partition $A : BC$, since $\langle 10 | \phi_+ \rangle = 0$. In this partition, the state is maximally entangled, hence $S(\rho_A) = 1$ and $S_L(\rho_A) = 1/2$. It is easy to check that the state can be rewritten as

$$|\psi\rangle_{ABC} = \frac{\sqrt{3}}{2} |0\rangle_B |\phi_+\rangle_{AC} + \frac{1}{2} |1\rangle_B |01\rangle_{AC} \tag{1.1.22}$$

and this is the Schmidt decomposition in the bi-partition $B : AC$. In this partition, however, the state is less than maximally entangled, and the entropies have values $S(\rho_B) = S(\rho_{AC}) = \frac{\log(4)}{4} - \frac{3}{2}\log\left(\frac{\sqrt{3}}{2}\right) \approx 0.81$ and $S_L(\rho_B) = 1 - 9/16 - 1/16 = 3/8$. The density matrix of C has the same entropy as B, due to the symmetry of the state under exchange of B and C, followed by a local unitary. Notice that the inequalities (1.1.18) and (1.1.20) are satisfied.

1.2 Entanglement in mixed states and separability criteria

In general, and always in practice, we are dealing with mixed states rather than pure states, due to the inevitable decoherence caused by the interaction of the observed system with the environment around it. It is then important to generalize the notions of entanglement and separability to mixed states of two systems X and Y. The direct generalization of the definition of separability for a pure state would be that a bipartite mixed state of X and Y is separable if it can be written as a product of two local density matrices

$$\rho_{XY}^{uncorr} = \rho_X \otimes \rho_Y. \tag{1.2.1}$$

Like in a separable pure state, the two subsystems don't have any information about each other, they are perfectly uncorrelated. However, this is not the commonly adopted definition of a separable state. The reason for this is that this set is not closed under LOCC operations.

Suppose that a person named Alice and another named Bob are each holding one of the systems X and Y in their respective quantum labs. These subsystems are perfectly uncorrelated, i.e. they are in a state of the form (1.2.1). Alice can roll a die, and if she gets the side labeled 1, with a probability 1/6, she performs a local operation on X, transforming the state into ρ_X^1. In this case, she contacts Bob through a classical channel, such as a telephone, and communicates to him the result, telling him to perform an operation on Y that will turn the subsystem into the state ρ_Y^1. For any other result of the die tossing, they do nothing. The state in this process becomes

$$\rho_{XY} = \frac{1}{6}\rho_X^1 \otimes \rho_Y^1 + \frac{5}{6}\rho_X \otimes \rho_Y. \tag{1.2.2}$$

This state cannot be written in the form (1.2.1) anymore. This shows that, unlike the set of pure separable states, the set of uncorrelated density matrices is not closed with respect to LOCC operations.

However, the whole set of density matrices

$$\rho_{XY}^{sep} = \sum_{i,j} q_{i,j} \rho_X^i \otimes \rho_Y^j. \tag{1.2.3}$$

is closed under LOCC [20]. For this reason, this is the generally accepted definition of separable density matrices.

By diagonalizing the local density matrices ρ_X^i and ρ_Y^j, we see that separable states can always be written as a mixture of pure states that are separable:

$$\rho_{XY}^{sep} = \sum_i p_i \left|\phi_X^i\right\rangle \left\langle\phi_X^i\right| \otimes \left|\chi_Y^i\right\rangle \left\langle\chi_Y^i\right|. \tag{1.2.4}$$

Every density matrix that cannot be written in a separable form is entangled. As a trivial example, a projector on a Bell state $\rho_{XY} = |\psi_+\rangle \langle\psi_+|$ cannot be decomposed in any of the above forms, and is therefore entangled according to this definition, as would be expected.

The sum in eq. (1.2.4) spans over a larger number of states than the dimension of the Hilbert space of X and Y, $d_X d_Y$. In general, the number of terms needed to decompose a separable state (1.2.3) is $(d_X d_Y)^2$ [21, 22]. This is one of the reasons why it is usually very hard to find whether a density matrix can be written in a separable form: it is in general not possible to find the pure states in eq. (1.2.4) by simply diagonalizing the density matrix. A separable density matrix can have entangled eigenstates. For example, while a projector on a Bell state is entangled, an even mixture of two Bell states $\rho_{XY} = 1/2 \left|\psi_+\right\rangle \left\langle\psi_+\right| + 1/2 \left|\psi_-\right\rangle \left\langle\psi_-\right|$ is separable, as we will prove later, even though it has two maximally entangled eigenvectors.

The difficulty to check whether a density matrix is separable or entangled is known as the *separability problem*. It is not as simple as for pure states, where just calculating the entropy of the subsystems gives the answer. However, some separability criteria are known. These are mathematical properties that distinguish separable states from entangled ones. We briefly list and discuss some of them.

1.2.1 Positive partial transpose

The operation of transposing a matrix is a positive map, in the sense that if a hermitian matrix $M = \sum_{i,j} |i\rangle \langle i| M |j\rangle \langle j|$ has all positive eigenvalues, the eigenvalues of the transposed matrix $M^T = \sum |i\rangle \langle j| M |i\rangle \langle j|$ are positive as well. If the positive matrix M describes a larger system

$$M_{XY} = \sum_{i,j,k,l} \Big(\langle i|_X \langle k|_Y M_{XY} |j\rangle_X |l\rangle_Y \Big) |i\rangle \langle j|_X \otimes |k\rangle \langle l|_Y, \tag{1.2.5}$$

the *partial transpose* with respect to Y

$$M_{XY}^{T_Y} = \big(\mathbb{I}_X + T_Y\big) M_{XY} = \sum_{i,j,k,l} \Big(\langle i|_X \langle l|_Y M_{XY} |j\rangle_X |k\rangle_Y \Big) |i\rangle \langle j|_X \otimes |k\rangle \langle l|_Y \tag{1.2.6}$$

is not always positive [23]. The operation of partial transposition does not change the trace of the matrix. For these reasons the operation of transposition T is said to be a positive, trace preserving (PT) map, but not a completely positive, trace preserving (CPT) map.

It is easy to see that all separable pure states are positive after partial transposition (PPT). The operation of partial transposition in the state (1.2.3) corresponds to transposing all the density matrices with the label Y. Because these are positive matrices, the operation makes the final matrix, that is also positive, another physical density matrix. Then, if a density matrix has some negative eigenvalues after the partial transposition, it must be entangled. This necessary criterion for separability was first shown by Peres [23] and is known as PPT, or Peres-Horodecki criterion.

In fact, the PPT condition is quite strict for separable states of low dimensions: Horodecki [24] proved that it is a necessary and also sufficient condition for separability in a Hilbert space of dimensions 2×2 and 2×3.

In general, any positive linear map that is not CP can provide a separability criterion. If we apply the positive map Λ to a density matrix such that

$$(\mathbb{I}_X \otimes \Lambda_Y)\rho_{XY} \tag{1.2.7}$$

has some negative eigenvalues, by the same argument that was used for transposition, we must conclude that the state ρ_{XY} is entangled [24]. The most general separability criterion is that ρ is separable if and only if it stays positive under all positive but not CP maps. The problem is that a complete characterization of this class of maps is not known, and the search for independent, nontrivial not CP maps is an active research field in linear algebra [25].

1.2.2 Entanglement witnesses

Since necessary and sufficient conditions for separability are difficult to obtain, practical entanglement detection is mostly via entanglement witnesses [26]. These witnesses are a class of observables $\mathcal{W}$ over the XY space that satisfy two conditions:

- $\mathcal{W}$ has at least one negative eigenvalue;
- the mean value of $\mathcal{W}$ over any separable pure state is positive: $\langle\phi_X|\langle\phi_Y|\mathcal{W}|\phi_X\rangle|\phi_Y\rangle \geq 0$ for all states $|\phi_X\rangle$ and $|\phi_Y\rangle$.

An example of an entanglement witness, if $d_X = d_Y = d$, is the swap operator

$$\Sigma = \sum_{i,j}^{d} |i\rangle\langle j| \otimes |j\rangle\langle i|, \tag{1.2.8}$$

that exchanges the state of X and Y. Here $\{|i\rangle, |j\rangle\}$ form a basis of the Hilbert space of the systems. The second condition is satisfied, because $\langle\phi_X|\langle\phi_Y|S|\phi_X\rangle|\phi_Y\rangle = |\langle\phi_Y|\phi_X\rangle|^2$. Also, any state that is anti-symmetric under the exchange of X and Y (such as the singlet state $|\psi_-\rangle$ if X and Y are qubits) is an eigenstate of S corresponding to the eigenvalue -1. S is therefore an entanglement witness [20].

If $\mathcal{W}$ satisfies both the above conditions and if the average of $\mathcal{W}$ over a state is negative:

$$\mathrm{Tr}[\mathcal{W}\rho_{XY}] < 0, \tag{1.2.9}$$

this is sufficient to show that ρ_{XY} is entangled.

The swap operator Σ, for instance, has a negative mean value over the singlet state $|\psi_-\rangle$ that is, of course, entangled. However, its mean value is positive over all

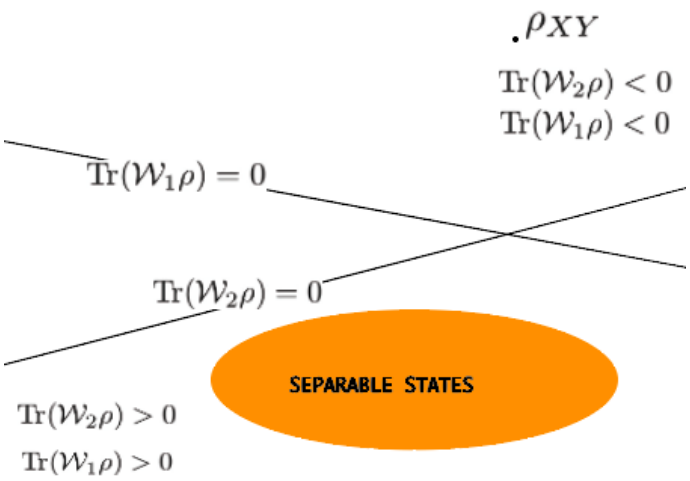

Figure 1.1: Entanglement witnesses as hyperplanes $\mathrm{Tr}[\mathcal{W}\rho] = 0$ in the space of the density matrices. Since the set of the separable states is convex, and it lies below the hyperplane ($\mathrm{Tr}[\mathcal{W}\rho_{sep}] \geq 0$), if a state is above the hyperplane, it must be entangled. Also, a state lies below all the hyperplanes described by all entanglement witnesses if and only if it is separable.

the other Bell states, showing that the condition (1.2.9) is only sufficient, and not necessary for entanglement.

The equation $\mathrm{Tr}(\mathcal{W}\rho_{XY}) = 0$ represents a hyperplane in the space of the density matrices, as shown in Fig. 1.1. By geometrical considerations, since the class of separable states is convex, it can be shown [26], that a state is entangled if and only if it satisfies inequality (1.2.9) for all the possible witnesses.

Entanglement witnesses are very useful in experiments, because they are explicitly observables. In this way, finding a negative average of this observable guarantees that the state is entangled, without having to perform a full tomography of the state.

1.2.3 Genuinely multipartite entanglement

For the most part of this thesis, even when we are dealing with a state of many particles, we are only concerned about bipartite entanglement, that is the entanglement that two parts of the system share with each other. However, more than two

quantum systems can be entangled with each other, not only in couples.

Consider a state of many parties, labeled $A, B, C....$ A genuinely multipartite entangled pure state is a state that is entangled in any bipartition:

$$|\psi\rangle_{ABC...} \neq |\phi\rangle_A |\phi\rangle_{BC...}, |\phi\rangle_{AB} |\phi\rangle_{C...}, |\phi\rangle_C |\phi\rangle_{AB...} \text{ etc.} \tag{1.2.10}$$

For example, for N qubits, the Greenberger-Horne-Zeilinger (GHZ) state

$$|GHZ\rangle = \frac{1}{\sqrt{2}} |000...0\rangle + \frac{1}{\sqrt{2}} |111...1\rangle \tag{1.2.11}$$

is entangled in any bipartition, and has then genuinely multipartite entanglement.

Another important class of multipartite pure entangled states is known as the class of Absolutely Maximally Entangled (AME) states. The special properties of such states is that they are maximally entangled for any bipartition that is chosen. It is easy to see that the GHZ state of three qubits is an AME state for such dimensions. However, it is not possible to build an AME state for any number of qubits. For four qubits, for instance, it was proven that AME states do not exist [27], and neither for more than eight qubits [28]. For seven qubits, it's not yet known whether AME states are possible. However, explicit examples were found [29] of AME states of five and six qubits. For instance, the state of five qubits

$$\begin{aligned} |AME\rangle \;=\; & \frac{1}{4}(|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\ & +|01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ & +|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\ & +|10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle). \end{aligned} \tag{1.2.12}$$

is maximally entangled in all possible bipartitions: writing the Schmidt decomposition of the state in a one-qubit basis, or in a two-qubits basis, you will always obtain a maximally entangled state.

The generalization of the concept of multipartite entanglement to mixed states is by imposing that the state is a mixture of genuinely multipartite entangled pure states. In other words, for any decomposition of a genuinely multipartite entangled

state ρ, there are no terms of the form

$$\rho = p_i \rho_X \otimes \rho_Y + ... \tag{1.2.13}$$

where X and Y represent *any* two groupings of the subsystems.

1.3 Entanglement measures

In this section I will introduce some entanglement measures, that is a number we can attach to any bipartite state to quantify its entanglement and, if possible, compare it to other states. Entanglement is essentially defined as a phenomenon that cannot be created by LOCC: the separable states have been defined as those that can be created via LOCC from a state that is uncorrelated. Then it is reasonable to expect that a measure of entanglement cannot grow under LOCC either.

In fact, the most commonly accepted definition of an entanglement measure is essentially through what is often called monotonicity axiom [30, 31]: a measure of entanglement is any quantity that does not grow, on average, under LOCC operations.

Because this is the generally accepted definition of an entanglement measure, often the term *entanglement monotone* is used as a synonym of entanglement measure. Some properties that follow from the monotonicity are

- For any separable state σ, $M(\sigma)$=0. If M is an entanglement monotone, it already must hold that M is a constant over the set of separable states. It is easy then, to put such constant to zero, if necessary;

- $M(\rho_{XY}) = M(U_X \otimes U_Y \rho_{XY} U_X^\dagger \otimes U_Y^\dagger)$, where U_X and U_Y are local unitaries. This holds because unitary operations are reversible.

Although monotonicity is the most important requirement for a function $M(\rho)$ to be considered an entanglement measure, there are other properties that are at least desirable for a measure. Some of them are

- $M(\sigma)$=0 if and only if σ is separable. This condition is not verified, for instance, for negativity and logarithmic negativity, that will be introduced later;
- M reduces to Von Neumann entropy for pure states. This requirement is also not always satisfied by many accepted measures;
- M is a convex function, i.e. $M(c_1\rho_1 + c_2\rho_2) \leq c_1 M(\rho_1) + c_2 M(\rho_2)$, if c_1 and c_2 are positive numbers.

We quickly now list and discuss some entanglement measures.

1.3.1 Entropies as entanglement measure

We saw, in sec. 1.1.2 that, if $|\psi_{XY}\rangle\langle\psi_{XY}|$ is a bipartite pure state, the Von Neumann entropy of the reduced density matrix ρ_X, we call it $E_S(|\psi_{XY}\rangle\langle\psi_{XY}|)$ is a measure of how uniform the Schmidt probabilities are. Then, if the Schmidt probabilities of a pure state $|\psi_{XY}\rangle\langle\psi_{XY}|$ majorize those of another state $|\phi_{XY}\rangle\langle\phi_{XY}|$, we must have $E_S(|\psi_{XY}\rangle\langle\psi_{XY}|) \leq E_S(|\phi_{XY}\rangle\langle\phi_{XY}|)$. From Nielsen's theorem, introduced in sec. 1.1.1, we conclude that the Von Neumann entropy of the reduced state is an entanglement monotone, therefore an entanglement measure for pure states. In fact, this is perhaps the most fundamental entanglement measure, and many other known measures, like entanglement of formation or relative entropy of entanglement, are a generalization of it to mixed states, like we will see in the following.

From an identical argument, the linear entropy, introduced in sec. 1.1.2 is also an entanglement monotone.

1.3.2 Negativity

The first entanglement measure for general mixed states that we present is the negativity after partial transposition, often simply called negativity [32]. We already introduced the concept of partial transposition (1.2.6), and the fact that separable states always stay positive under such operation. Negativity is defined as the magnitude of the sum of all the negative eigenvalues of the matrix obtained from applying the operation of partial transpose to the state. This function satisfies the

monotonicity axiom [32], and is zero for all separable states. For all entangled states of dimension 2×2 or 2×3, it is strictly positive, while for higher dimensions there exist some entangled states that have zero negativity [24].

Given a generic bipartite density matrix ρ_{XY}, the negativity can be written as

$$N_{X:Y}(\rho_{XY}) = \sum_{i=0}^{d-1} \frac{|\lambda_i| - \lambda_i}{2} = \frac{\sum_{i=0}^{d-1} |\lambda_i| - 1}{2}, \tag{1.3.1}$$

where λ_is are the eigenvalues of the partially transposed matrix ρ_{XY}^{PT}. The second equality holds because the operation of partial transposition does not affect the trace, that is then equal to 1.

Eq. (1.3.1) is often written as

$$N_{X:Y}(\rho_{XY}) = \frac{||\rho^{PT}|| - 1}{2} \tag{1.3.2}$$

where $||M|| = \mathrm{Tr}[\sqrt{M^\dagger M}]$ is the trace norm of the generic matrix M. Like any norm, $||M||$ satisfies the condition $||cM|| = |c|||M||$, for any scalar c, and the triangle inequality $||M+M'|| \leq ||M||+||M'||$. Using these inequalities, we see that negativity is a convex function:

$$N_{X:Y}\left(\sum_i p_i \rho_i\right) \leq \sum_i p_i N_{X:Y}(\rho_i) \tag{1.3.3}$$

An advantage of N as a measure is that it is very easy to compute, at least if the dimensions of X and Y are not high, so that the diagonalization becomes difficult.

Example: negativity of a Bell state

Consider a Bell state $|\phi_+\rangle = 1/\sqrt{2}(|00\rangle+|11\rangle)$ of two qubits X and Y. The projector on this state $|\phi_+\rangle \langle\phi_+|$ is

$$|\phi_+\rangle \langle\phi_+| = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11| + |00\rangle \langle 11| + |11\rangle \langle 00|), \tag{1.3.4}$$

so the partial transposition

$$|\phi_+\rangle\langle\phi_+|^{T_X} = \frac{1}{2}(|00\rangle\langle 00| + |11\rangle\langle 11| + |01\rangle\langle 10| + |10\rangle\langle 01|), \tag{1.3.5}$$

which in the canonical basis $\{|00\rangle, |01\rangle, |10\rangle, |11\rangle\}$ is

$$|\phi_+\rangle\langle\phi_+|^{T_X} = \begin{pmatrix} \frac{1}{2} & 0 & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ 0 & \frac{1}{2} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{2} \end{pmatrix} \tag{1.3.6}$$

This matrix has three positive eigenvalues $1/2$ and one negative eigenvalue $-1/2$. Hence, the negativity of this state is $1/2$. Since all the Bell states can be obtained from $|\phi_+\rangle$ via local unitary transformations, they all have negativity $1/2$. This is the largest negativity that can be achieved by a state of two qubits, or any density matrix of rank two.

Example: negativity of a Bell-diagonal state

We now calculate the negativity of a class of states that will be very useful in the rest of this thesis, the class of the Bell diagonal states.

These are states of two qubits that are obtained by mixing projectors on the four states of the Bell basis:

$$\rho_{XY} = A|\phi_+\rangle\langle\phi_+| + B|\phi_-\rangle\langle\phi_-| + C|\psi_+\rangle\langle\psi_+| + D|\psi_-\rangle\langle\psi_-|, \tag{1.3.7}$$

where A, B, C and D are probabilities, i.e. positive numbers that sum to 1. If we transpose with respect to X, the matrix we obtain in the canonical basis is

$$\rho_{XY}^{T_X} = \frac{1}{2}\begin{pmatrix} A+B & 0 & 0 & C-D \\ 0 & C+D & A-B & 0 \\ 0 & A-B & C+D & 0 \\ C-D & 0 & 0 & A+B \end{pmatrix}. \tag{1.3.8}$$

This matrix has eigenvalues $\{\frac{A+B+C-D}{2}, \frac{A+B-C+D}{2}, \frac{A-B+C+D}{2}, \frac{-A+B+C+D}{2}\}$. If $A = 1$, and so $B = C = D = 0$, we recover the negativity of $|\phi_+\rangle$ in the previous example. Each of these eigenvalues are positive, unless one of the coefficients is larger than $1/2$, given that their sum has to be 1. If we call $\lambda_1 = \max(A, B, C, D)$ the largest eigenvalue of the original density matrix, we get that $N(\rho) = \max(\lambda_1 - 1/2, 0)$. Since negativity under partial transposition is both necessary and sufficient for entanglement of two qubits, we have shown that a Bell diagonal state is entangled if and only if its largest eigenvalue is larger than 0.5.

1.3.3 Logarithmic negativity

Another entanglement monotone is the logarithmic negativity [32, 33], defined as

$$L_{X:Y}(\rho) = \log_2 \|\rho_{XY}^{PT}\| = \log_2(2N_{X:Y} + 1). \tag{1.3.9}$$

Since $L_{X:Y}(\rho)$ is a monotonic function of the negativity of the state, if $N_{X:Y}$ does not increase under LOCC, neither does $L_{X:Y}$. Like negativity, its value is zero when applied to a separable state. For example, in a Bell state of two qubits, its value is 1.

Differently from negativity and many other measures, however, this function is not convex.

1.3.4 Relative entropy of entanglement

The relative entropy between the density matrix ρ and another σ, of the same dimension, is defined as

$$S(\rho||\sigma) = -\operatorname{Tr}(\rho \log \sigma) - S(\rho) = \operatorname{Tr}[\rho(\log \rho - \log \sigma)] \tag{1.3.10}$$

where $S(\rho)$ is the Von Neumann entropy, defined as in eq. (1.1.17). This function is nonnegative, for any pair of density operators, and its value is zero if and only if $\rho = \sigma$. In this sense, it can be considered as a measure of the distance between the two density matrices. However, it is not strictly a distance, not being in general

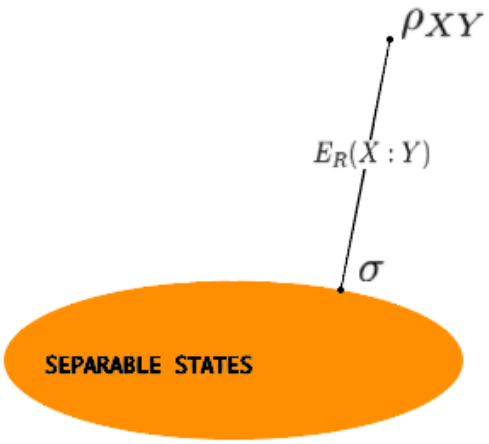

Figure 1.2: Graphic representation of the relative entropy of entanglement of a state ρ. The state σ is the closest separable state to ρ as measured by the quantum relative entropy.

symmetric: $S(\rho\|\sigma) \neq S(\sigma\|\rho)$.

Still, this interpretation as a pseudo-distance gives a useful way to measure the entanglement of the state. Since the set S of separable states is convex, we expect that the more a state is entangled, the farther it lies from the set. We can, then, define an entanglement measure as

$$E_R(\rho) = \inf_{\sigma \in S} S(\rho\|\sigma), \tag{1.3.11}$$

that is, the distance from the closest separable state. This idea was first introduced in [31], where it was also proven that $E_R(\rho)$ it is a convex entanglement monotone.

Other distance-based entanglement measures are possible and have been proposed [34, 35]. The idea of entanglement as a distance is particularly useful because it allows comparing, directly and quantitatively, entanglement with other forms of classical and quantum correlations, as we will see later. However, one disadvantage of this measure is that the minimization over the set of separable states is in general difficult to perform, analytically or numerically, making it hard to compute.

1.3.5 Other entanglement measures

Among many entanglement monotones that are known (see [36] for a review), I would like to mention briefly some that have been useful for my research.

Optimal singlet fraction

The optimal singlet fraction [37], also called fidelity [30] describes how similar a state ρ is to a maximally entangled state

$$\mathcal{F}(\rho) = \max_{\Phi}(\langle\Phi|\,\rho\,|\Phi\rangle), \tag{1.3.12}$$

where the maximum is taken over all maximally entangled states $|\Phi\rangle$.

Concurrence

For a density matrix of two qubits ρ, the *concurrence* [38] is defined as

$$C(\rho) = \max(0, \lambda_1 - \lambda_2 - \lambda_3 - \lambda_4), \tag{1.3.13}$$

where λ_i is the i^{th} largest eigenvalue of the matrix $M = \sigma_Y \otimes \sigma_Y \rho^* \sigma_Y \otimes \sigma_Y$. Here,

$$\sigma_Y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix} \tag{1.3.14}$$

is the Pauli matrix.

Entanglement of formation

The entanglement of formation [30] is defined as the smallest average Von Neumann entropy among all the possible pure state decompositions of the state $\rho = \sum_i p_i\,|\psi_i\rangle\langle\psi_i|$:

$$E_F = \inf_{\{p_i,\psi_i\}} \left(\sum_i p_i E_S(|\psi_i\rangle\langle\psi_i|) \right). \tag{1.3.15}$$

For a state of two qubits, the entanglement of formation is univocally given by concurrence [39].

1.3.6 Operational measures of entanglement

We have given the fundamental axiom that all entanglement measures need to satisfy. However, there is a different approach to define the amount of entanglement stored in a state, which is considering it as a resource. The question of how much a state is entangled, then, translates into how much of this resource can be extracted from the state, or how much is needed to create it. We now introduce two entanglement measures that directly stem from this idea: distillable entanglement and entanglement cost [31].

Distillable entanglement

The distillable entanglement describes the minimum number of copies of the state, n, that are needed in order to create m copies of maximally entangled states in the asymptotic case where $n \to \infty$. We will describe the process of entanglement distillation in sec. 2.3.1. However, roughly speaking, it is the process of extracting some maximally entangled pairs from more non-maximally entangled pairs only via LOCC.

Entanglement cost

Entanglement cost answers the inverse question to distillable entanglement. This question is how many copies n of a maximally entangled state do we need in order to create m copies of the state only with LOCC. The asymptotic rate $\lim_{n\to\infty} m/n$ is the entanglement cost E_C.

Operational meaning of entanglement monotones

The distillable entanglement, the entanglement cost, or any measure of entanglement that is defined via the protocols that can be realized from the state are often difficult to compute. Axiomatic measures, on the other hand, only satisfy the monotonicity

condition, and don't always have a clear operational meaning. However, they are usually easier to compute. For this reason, it is interesting to see what the number described by entanglement monotones mean, how it can be used as a resource. Among others, we list some operational properties associated with the entanglement monotones that were listed above:

- negativity, singlet fraction and concurrence of a state all provide an upper bound to the accuracy that can be reached when performing a teleportation protocol [37, 40];
- Logarithmic negativity provides an upper bound to distillable entanglement;
- The entanglement of formation of infinite number of copies of the system is equal to the entanglement cost of the state [41]: $E_C(\rho) = \lim_{n\to\infty} \frac{1}{n} E_F(\rho^{\otimes n})$.

1.4 Correlations between quantum systems

Entanglement is probably the most prominent and the most studied among the forms of correlations that two quantum systems can share. However, other forms of correlations are known. Some of them are classical (two quantum systems can share some mutual information like classical systems do). Others are impossible to attain in a classical system. In this section, we will introduce quantum discord, a form of purely quantum correlation, more general than entanglement, that can be found in mixed states. Later, we will also define the correlation tensor, a way to describe the correlations between local measurements on different parties.

1.4.1 Mutual Information

Mutual information quantifies any possible form of correlations that two systems can share.

In this section, we first introduce very briefly some elements of classical information theory needed to understand the concept of mutual information between two classical systems, and then discuss how such definition can be generalized to

quantum systems. A much more extensive treatment of the classical case can be found in [42], while for the generalization to quantum systems and the definition of quantum discord, you can see [1].

Classical mutual information

In classical information theory, mutual information describes how much information one system stores about another.

Suppose a source generates randomly a string of values corresponding to an alphabet, according to the probability distribution $p(a)$. If another source also emits a message in the form of a string with values according to probability $q(a)$, the relative entropy, or Kullback-Leibler distance between the probability distributions q and p is defined as

$$D(p||q) = -H(p) - \sum_a p(a)\log(q(a)), \tag{1.4.1}$$

where $H(p)$ is the Shannon entropy of the distribution p:

$$H(\{p_a\}) = -\sum p_a \log_2(p_a). \tag{1.4.2}$$

Since the Kullback-Leibler distance is always strictly positive unless the two probability distributions are identical, it can be interpreted as the distance between the distributions p and q.

Now, suppose that the message (a random variable) is divided into two messages (two random variables), X and Y, that are encoded as the pair $\{x, y\}$. The probability of having the pair of values $\{x, y\}$ is $p(x, y)$, while the probability of having only the value x of the variable X is given by $p(x) = \sum_y p(x, y)$, and the same for Y, $p(y) = \sum_x p(x, y)$. On the other hand, if we already measured X and found its value to be $\bar{x}$, the conditional probability, or the probability of Y having value $\bar{y}$ when X has $\bar{x}$, is

$$p(y = \bar{y}|x = \bar{x}) = \frac{p(\bar{x}, \bar{y})}{\sum_y p(\bar{x}, y)} = \frac{p(\bar{x}, \bar{y})}{p(\bar{x})}. \tag{1.4.3}$$

Suppose that X and Y are two completely independent variables. Then, we must

have that $p(x)p(y) = p(x, y)$, for any values of x and y.

Therefore, if two variables are correlated, the Kullback-Leibler distance between the distribution $p(x, y)$ and the distribution $q(x, y) = p(x)p(y)$ must be strictly positive. This is how the mutual information between X and Y is defined:

$$I(X;Y) = D\left(p(x,y)||p(x)p(y)\right) = H(X) + H(Y) - H(X,Y), \tag{1.4.4}$$

The second equality is obtained by the definition of relative entropy, eq. (1.4.1).

Another way of writing the mutual information is by defining the *conditional entropy* of the bipartite system:

$$H(Y|X) = -\sum_x p_x H(Y|x) = -\sum_x p_{x,y} \log(p(y|x)) = H(X,Y) - H(X), \tag{1.4.5}$$

The mutual information, then, can be re-written in terms of the conditional entropy:

$$I(X;Y) = H(X) - H(X|Y). \tag{1.4.6}$$

Holevo bound

The Holevo theorem [43] describes the amount of classical mutual information that can be extracted from a quantum system. Suppose that Alice wants to encode a message in her quantum system A. To do this, she prepares a different quantum state ρ_x according to the probability p_x of a classical variable X. Then Bob receives the system A in the state $\rho = \sum_x p_x \rho_x$, and tries to decode the information on the variable X by taking POVM measurements (see Appendix) on A. The measurement result defines a random variable Y.

The Holevo theorem states that, in this scenario, the mutual information between the variables Y and X is bound from above:

$$I(Y : X) \leq S(\rho) - \sum_x p_x S(\rho_x), \tag{1.4.7}$$

where S represents the Von Neumann entropy.

If A is a system composed of N qubits, the Von Neumann entropy is at most

N. Then we have $I(Y:X) \leq N$, so a consequence of the Holevo theorem is that at most N bits of classical information can be encoded in a system of N qubits.

1.4.2 Mutual information between quantum systems

Given a density matrix of two quantum systems X and Y, ρ_{XY}, the most natural generalization of the classical definition of mutual information is to take the distance, expressed by the relative entropy, between the state ρ_{XY} and the closest state uncorrelated state. It turns out that the closest such state is simply given by the tensor product of the reduced density matrices $\rho_X \otimes \rho_Y$. We therefore have

$$I(X;Y) = S(\rho_{XY}|\rho_X \otimes \rho_Y) = S(\rho_X) + S(\rho_Y) - S(\rho_{XY}), \tag{1.4.8}$$

where $S(\rho|\sigma)$ is the quantum relative entropy and $S(\rho)$ the Von Neumann entropy of the density matrix. This is the mutual information between two systems X and Y, and it quantifies all the forms of correlations that the systems have about each other. However, there are other ways of extending the classical definitions to quantum systems. Whenever a measurement is conducted on a quantum system, the results are typically random. We can use the resulting probability distribution as the classical distribution in the previous section, and check how much a measurement on a subsystem says about the other subsystem.

Suppose that Bob performs a projective measurement of the observable O_Y that is diagonal on the basis $\{|\psi_0\rangle, ... |\psi_i\rangle, ... |\psi_{d_Y-1}\rangle\}$ (see Appendix). If he measures the i^{th} outcome, the state of X after the measurement on Y is

$$\rho_{X|i} = \frac{\langle\psi_i| \rho_{XY} |\psi_i\rangle}{\mathrm{Tr}(|\psi_i\rangle\langle\psi_i| \rho_{XY})} \tag{1.4.9}$$

and the probability of this measurement outcome is given by $p_i = \mathrm{Tr}(|\psi_i\rangle\langle\psi_i| \rho_{XY})$. The 'quantum-classical' conditional entropy can therefore be defined, similarly to (1.4.5) as

$$S_{qc}(X|Y) = \sum_i p_i S(\rho_{X|i}), \tag{1.4.10}$$

where $S(\rho_{X|i})$ is the Von Neumann entropy of the state (1.4.9). This definition depends on the basis of the particular observable O_Y, and is not symmetric if the roles of X and Y are exchanged.

An alternative definition of mutual information between two subsystems, then, is by rewriting Eq. (1.4.6)

$$J(X|Y) = S(\rho_X) - \sup_{O_Y} S_{qc}(X|Y), \tag{1.4.11}$$

The first element on the right hand side is still the Von Neumann entropy of the reduced state of X. The second term is the maximum value of the conditional entropy (1.4.14) over all possible measurement bases.

These two definitions of mutual information between two quantum systems are not always equivalent. In fact $J(X|Y)$ describes the classical information that the two systems share. This brings us to the definition of quantum discord.

1.4.3 Quantum discord

The difference between the two definitions of mutual information above, that are perfectly equivalent in classical theories, was first noted in [44, 45].

Consider the state of two qubits

$$\rho_{XY} = \frac{1}{3}|0\rangle\langle 0| \otimes |0\rangle\langle 0| + \frac{1}{3}|1\rangle\langle 1| \otimes |1\rangle\langle 1| + \frac{1}{3}|+\rangle\langle +| \otimes |+\rangle\langle +|, \tag{1.4.12}$$

where $|+\rangle = 1/\sqrt{2}(|0\rangle + |1\rangle)$. The Von Neumann entropy of the whole state is $S(\rho_{XY}) = \frac{\log(3)}{3} - \frac{1}{3}\left(2-\sqrt{2}\right)\log\left(\frac{1}{6}\left(2-\sqrt{2}\right)\right) \sim 1.32$.

The reduced states of X and Y have the same spectrum $\left\{\frac{2}{3}, \frac{1}{3}\right\}$, then $S(\rho_X) = S(\rho_Y) = \frac{2}{3}\log\left(\frac{3}{2}\right) + \frac{\log(3)}{3} \sim 0.92$. Thus, the mutual information between X and Y in this state is

$$I(X;Y) = S(\rho_X) + S(\rho_Y) - S(\rho_{XY}) \sim 0.52. \tag{1.4.13}$$

Now, we calculate the quantum-classical conditional entropy between X and Y

$$S_{qc}(X|Y) = \sum_i p_i S(\rho_{X|i}). \tag{1.4.14}$$

given a generic measurement over any basis of qubit Y.

It is quite intuitive that the maximum conditional entropy is obtained when measuring Y over an observable diagonal in the canonical basis, i.e. $|0\rangle$ and $|1\rangle$. This was also shown in the original article [44]. Then the two outcomes, 0 and 1 have the same probability, $1/2$ of being measured on this state. Also, the states of X obtained by measuring the value 0 on Y has Von Neumann entropy: $S(\rho_X^0) = -\frac{1}{6}\left(3-\sqrt{5}\right)\log\left(\frac{1}{6}\left(3-\sqrt{5}\right)\right) - \frac{1}{6}\left(3+\sqrt{5}\right)\log\left(\frac{1}{6}\left(3+\sqrt{5}\right)\right) \sim 0.55$, and $S(\rho_X^1)$ is identical. Then, according to the definition (1.4.11), $S(X|Y) = 1/2S(\rho_X^0)+1/2S(\rho_X^1) = S(\rho_X^0)$. Then we have

$$J(X|Y) = S(\rho_X) - S(\rho_X|\rho_Y) \simeq 0.92 - 0.55 = 0.37. \tag{1.4.15}$$

The difference between the two mutual informations (1.4.13) and (1.4.15) is due to the fact that a measurement performed on Y affects the system X as well. This is impossible with a classical system. The only states that have zero discord are of the form

$$\rho_{XY}^{cl} = \sum_{i=0}^{d_y-1} p_i \rho_i^X \otimes |\psi_i\rangle\langle\psi_i| \,. \tag{1.4.16}$$

where $\langle\psi_i|\psi_i\rangle = \delta_{ij}$. For this class of states there exists at least a measurement on Y, the one taken on a basis that includes the vectors $|\psi_i\rangle$, that does not perturb the state of X. These are often called classical states (or quantum-classical states, as opposed to the classical-quantum ones where the roles of X and Y are interchanged).

The quantity $I(X;Y) - J(X|Y)$ is called quantum discord $D(X|Y)$. Discord is never negative for any state [46]. Also, it is not symmetrical under the exchange of X and Y: $D(X|Y) \neq D(Y|X)$. Notice that the discorded state that was described in this section is separable (and can therefore be made by LOCC). Discord is a weaker property than entanglement: all entangled states are discorded, but not vice versa.

The existence of discord in separable states means that entanglement does not take into account all possible forms of quantum correlations. The study of the properties of discord and discorded states is very broad (see [47] for a review) and some protocols have been found suggesting that discord is a useful resource in quantum communication protocols [48–52]

Relative entropy of discord

An alternative measure of discord is the relative entropy of discord [53, 54]. The idea is similar to that of the relative entropy of entanglement, introduced in Sec. 1.3.4, where entanglement was measured as a distance (in terms of relative entropy) between the state and the closest separable state. Discord can also be measured as the distance from the closest possible classical state, of the form (1.4.16).

$$D_R(\rho_{XY}) = \inf_{\chi \in \mathcal{C}} S(\rho_{XY}|\chi), \tag{1.4.17}$$

where χ is a state belonging to the set of classical states $\mathcal{C}$. This measure is often called *relative entropy of discord*. Recall that mutual information was also defined in terms of relative entropy from the closest uncorrelated state. An illustration of correlations as relative entropies is shown in Fig. 1.3. In general, any form of correlation can be defined as a distance from the set of states that do not present this property. The beauty of such a unified view is that all forms of correlations can be compared with one another, and that can be naturally extended to multipartite systems. The search for other measures of quantum correlations that allow a unified definition of quantum correlations is indeed very interesting [55, 56].

1.4.4 Correlation Tensor

The correlation tensor is a different way of describing the presence of correlations between quantum systems. I will first introduce the Bloch vector of a qubit state and observable, and then proceed to generalize this concept to a multi-partite system, presenting the correlation tensor.

I will then discuss a physical interpretation of this tensor as related to the corre-

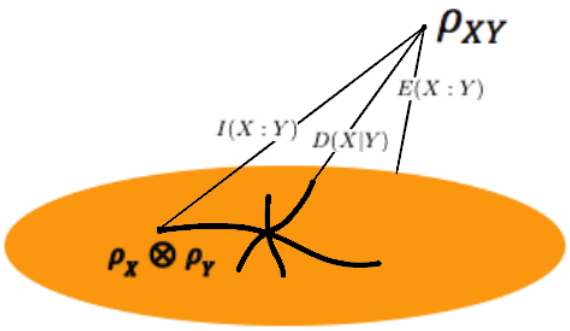

Figure 1.3: An illustration of the idea of correlations as distances. The set of uncorrelated states (represented by the black lines) lies inside both the classical and the separable sets. Hence the mutual information, defined as the minimum distance to the set is the most inclusive correlation between two parties, and can be generalized similarly to many parties. The discord can be defined as the distance from the set of classical correlated states, while entanglement is measured as the distance from the larger separable set.

lations that are obtained by measuring local observables in a multi-partite quantum system.

Bloch vector of a qubit state

Consider a general density matrix of a qubit. This is a hermitian, positive-definite 2×2 matrix with trace equal to 1. As such, it depends on three real numbers, and it can always be written as

$$\rho = \begin{pmatrix} \frac{1+a}{2} & \frac{c-id}{2} \\ \frac{c+id}{2} & \frac{1-a}{2} \end{pmatrix}. \tag{1.4.18}$$

By defining the 3-dimensional Bloch vector $\vec{s} = \{a, b, c\}$, the matrix above can be written as

$$\rho = \frac{\sigma_0}{2} + \frac{\vec{s} \cdot \vec{\sigma}}{2}. \tag{1.4.19}$$

where σ_0 is the identity matrix

$$\sigma_0 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \tag{1.4.20}$$

and $\vec{\sigma} = \{\sigma_1, \sigma_2, \sigma_3\}$ are the three Pauli matrices

$$\sigma_1 \equiv \sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.4.21}$$

$$\sigma_2 \equiv \sigma_y = \begin{pmatrix} 0 & -i \\ i & 0 \end{pmatrix}, \tag{1.4.22}$$

$$\sigma_3 \equiv \sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.4.23}$$

Each component of the Bloch vector can be evaluated as $s_i = \mathrm{Tr}(\rho\sigma_i)$.

Its norm, as is easy to check by taking the square of equation (1.4.19), is at most one $|\vec{s}|^2 \leq 1$, and the equality only holds for pure states. The Bloch vector maps the space of density matrices of a qubit into a sphere of radius one in the three-dimensional cartesian space. As is easy to show, the Bloch vectors of two orthogonal pure states are parallel and opposite to each other.

Bloch vector of an observable of a qubit

Any projective measurement on a qubit has at most two outcomes. Without loss of generality, assume that these eigenvalues are ± 1. Then any observable can be written as

$$\mathcal{O} = |m_+\rangle\langle m_+| - |m_-\rangle\langle m_-| = \frac{\sigma_0}{2} + \frac{\vec{m}\cdot\vec{\sigma}}{2} - \frac{\sigma_0}{2} + \frac{\vec{m}\cdot\vec{\sigma}}{2} = \vec{m}\cdot\vec{\sigma} \tag{1.4.24}$$

where $\vec{m}$ is the Bloch vector of $|m_+\rangle$ and we used the fact that the two orthogonal states $|m_+\rangle$ and $|m_-\rangle$ have opposite Bloch vectors.

Correlation tensor of many qubits

We are now ready to define the correlation tensor of a state of many qubits. As the four Pauli matrices form a basis of the space of density matrices of one qubit, the

space of N qubits is spanned by the 4^N matrices

$$\sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes ... \otimes \sigma_{\mu_2}, \tag{1.4.25}$$

where the indeces μ_i run from 0 to 3.

Therefore, a general density matrix of N qubits can be written as

$$\rho_N = \sum_{\mu_1,\mu_2,...,\mu_N=0}^{3} T_{\mu_1,\mu_2,...,\mu_N} \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes ... \sigma_{\mu_N}. \tag{1.4.26}$$

Here, we have already defined the extended correlation tensor of the state

$$T_{\mu_1,\mu_2,...,\mu_N} = \mathrm{Tr}\left(\rho \sigma_{\mu_1} \otimes \sigma_{\mu_2} \otimes ... \otimes \sigma_{\mu_N}\right). \tag{1.4.27}$$

The normalization of the density matrix implies that $T_{0,0,...,0} = 1$.

To understand the meaning of this tensor, imagine that N observers perform each a measurement over their respective qubits. Each of them obtains a result that is either 1 or -1. The correlation function between the outcomes of the measurements is defined as the average of the product of the outcomes. Using the parametrization of the one-qubit operators as Bloch vectors, this reads:

$$C_{\vec{m}_1,...,\vec{m}_N} = \mathrm{Tr}\left(\rho_N \vec{m}_1 \cdot \vec{\sigma}^{(1)} \otimes ... \otimes \vec{m}_N \vec{\sigma}^{(N)}\right). \tag{1.4.28}$$

From the form of the density matrix (1.4.26), the last expression can be also written as

$$C_{\vec{m}_1,...,\vec{m}_N} = \sum_{k_1=1}^{3} ... \sum_{k=1}^{3} T_{k_1,...,k_N} (\vec{m_1})_{k_1} ... (\vec{m_N})_{k_N} = \hat{T} \cdot \vec{m}_1 \otimes ... \otimes \vec{m}_N, \tag{1.4.29}$$

where $(\vec{m_i})_{k_i}$ is the k_i^{th} component of the Bloch vector associated with the measurement on the i^{th}. The last expression denotes the scalar product, in the $3N$-dimensional cartesian space, between the components of the correlation tensor, that we may call N-partite correlation tensor, and the measurement Bloch vectors.

If the N-partite correlation tensor of a state is zero, any set of local measurements

will always be uncorrelated. From a similar derivation, we would see that the tensor $\hat{T}_M = T_{k_1,\ldots k_M,0\ldots0}$, with $M \leq N$, that we may also call $M-$partite correlation tensor, describes the correlation functions between local measurements performed on the first M qubits.

2. Introduction: quantum communication

Quantum communication studies the strategies through which a communication process can be performed by using quantum states and operations. The communication can be classical, occurring through classical channels such as internet or a telephone, or quantum, where a part of the quantum system is sent.

In this chapter, some protocols for quantum communication are presented. First, we present the formalism of Kraus operators for quantum channels, then we describe some famous examples of quantum communication involving entanglement. Finally, we review ways in which entanglement can be communicated, distilled, or distributed.

2.1 Quantum channels

Quantum channels are the most general quantum operations because, when applied on a part of the system described by a generic density matrix, give as a result another density matrix. In order to do this, a map Φ between two matrix spaces $\mathcal{L} \to \mathcal{M}$ must be:

- linear i.e $\Phi(aA + bB) = a\Phi(A) + b\Phi(B)$ for any operators A and B in $\mathcal{L}$ and c-numbers a and b;

- completely positive (CP), i.e. $(\mathbb{I}_X \otimes \Phi_Y)A_{XY}$ has all positive eigenvalues when applied to a positive operator $A_{XY} \in \mathcal{H} \otimes \mathcal{L}$;

- trace-preserving (CPT), i.e. $\mathrm{Tr}\left(\Phi(A)\right) = \mathrm{Tr}(A)$ for any $A \in \mathcal{L}$.

For example, the operation of trace is a quantum channel, having all the three properties that we listed. Transposition, on the other hand, is not, because, even though it's both linear and trace-preserving, it is not completely positive.

A way of representing any completely positive, linear map is in terms of matrices, known as Kraus operators, [57]:

$$\Phi(\rho) = \sum_n K_n \rho K_n^\dagger. \tag{2.1.1}$$

For this map to be trace-preserving, we need one additional condition

$$\sum_n K_n^\dagger K_n = \mathbb{I}. \tag{2.1.2}$$

The Kraus representation for a given channel is not unique. However, for any channel, it is always possible to find a minimal Kraus representation that includes $d_\mathcal{L} \times d_\mathcal{M}$ operators. If, in addition, the channel acting on an identity matrix gives another identity matrix, the channel is said to be unital.

Kraus operators between the same space describe a unitary evolution of the system in a larger Hilbert space [58]. To see this, consider a systems A and ancilla B. They are initially uncorrelated, so that the initial density matrix can be written as $\rho(0) = \rho_A \otimes \rho_B$. If at time $t = 0$ they start interacting with Hamiltonian H_{AB}, the state at time t is given by

$$\rho(t) = U_t \rho_A \otimes \rho_B U_t^\dagger, \tag{2.1.3}$$

where U_t is the unitary matrix $e^{-i\frac{H_{AB}t}{\hbar}}$.

By tracing out the system B, on the eigenbasis of $\rho_B = \sum \lambda_i |i\rangle\langle i|$, we see that the state of A can be written as

$$\rho_A(t) = \sum_{i,j} \langle i| U \sqrt{\lambda_i} |j\rangle \, \rho_A \, \langle j| U^\dagger \sqrt{\lambda_i} |i\rangle . \tag{2.1.4}$$

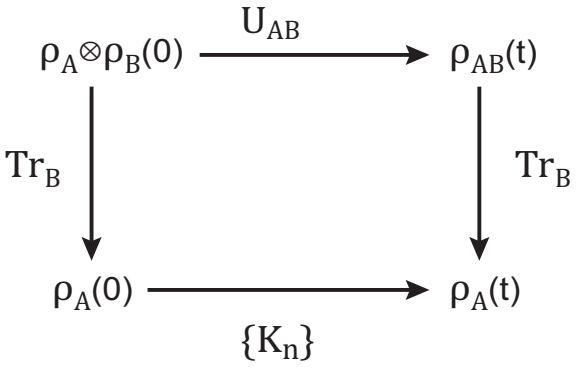

Figure 2.1: A quantum channel, represented by Kraus operators K_n, describes a unitary interaction of the system with an ancillary environment B.

This is similar to eq. (2.1.1), where $K_n = K_{ij} = \langle i| U \sqrt{\lambda_i} |j\rangle$. It is easy to check that these Kraus operators also satisfy the condition in eq. (2.1.2).

The inverse is also true: a linear CP map on the matrix space $\mathcal{L}$, or equivalently, a set of Kraus operators, can always be described by a unitary operation on a larger space, according to the Stinespring dilation theorem [59]. This idea is illustrated in fig. 2.1. From a physical perspective, then, quantum channels describe the interaction of the system with its environment. In this sense, they describe the noise that affects a quantum particle traveling between two places.When the coupling with the environment is strong, the state of the system can be greatly disrupted, resulting in a strong quantum noise.

2.1.1 Entanglement breaking channels

Entanglement breaking channels (EBC) are a class of quantum channels, that break the entanglement of any system that crosses them with the rest of the world. This means that, for any state of a bipartite system of particle X, of dimension d_X and Y, of dimension d_Y, by sending Y through a EBC channel, the final state will always be separable in the partition $Y : X$.

In [60] some equivalent conditions for EBC are derived. In particular, we are mostly interested in three necessary and sufficient conditions for characterizing them.

Given a CPT map Φ, the following conditions are equivalent:

- Φ is entanglement breaking.
- Φ breaks the entanglement of any maximally entangled state. That is, $(\mathbb{I} \otimes \Phi) |\phi^+\rangle \langle\phi|$ is separable, where $|\phi\rangle$ is a maximally entangled state:

$$|\phi\rangle = U_X \otimes U_Y \frac{1}{\sqrt{d_Y}} \sum_j |j\rangle |j\rangle, \tag{2.1.5}$$

  where $\{|j\rangle\}$ is a local basis.
- Φ can be represented by rank-1 Kraus operators: $K_n = |v_n\rangle \langle w_n|$.

The last condition implies that the channel applies a projective measurement on the subsystem Y that crosses it:

$$\Phi(\rho_Y) = \sum_n |v_n\rangle\langle v_n| \operatorname{Tr}\Big[|w_n\rangle\langle w_n| \rho_Y \Big]. \tag{2.1.6}$$

If Y is a part of a bipartite system, a representation of the action of Φ on the subsystem Y is

$$(\mathbb{I}_X \otimes \Phi_Y)\rho_{XY} = \sum_n \operatorname{Tr}_Y(|w_n\rangle \langle w_n| \rho_{XY}) \otimes |v_n\rangle \langle v_n|_Y, \tag{2.1.7}$$

where $\{|v_n\rangle\}$ and $\{|w_n\rangle\}$ are two generally over-complete sets in the Hilbert space of Y.

2.1.2 One-qubit quantum channels

We now review some notable examples of noisy channels acting on a single qubit. All the channels that we describe here depend on one parameter, that physically represents the strength of the coupling with the environment, or the time spent in the channel. We renormalize this parameter such that it is in the range $[0, 1]$ in all our examples. We give the minimal representation, in terms of four or less Kraus operators.

Dephasing channel

This channel describes a process in which a quantum bit has a certain probability $\frac{\delta_{\text{ph}}}{2}$ of adding a phase of π the off-diagonal terms of its density matrix. Otherwise, with probability $1-\frac{\delta_{\text{ph}}}{2}$, it is unaltered by the channel. This is a simple description of physical processes in which the interaction with the environment causes a loss of coherence in one privileged direction.

The Kraus operators of the dephasing channel take the form

$$K_0^{(\text{ph})} = \sqrt{1-\frac{\delta_{\text{ph}}}{2}}\sigma_0, \qquad K_1^{(\text{ph})} = \sqrt{\frac{\delta_{\text{ph}}}{2}}\,\sigma_z. \tag{2.1.8}$$

This channel is unital. where we used that $\sigma_z^2 = \sigma_0$.

To see if this channel is entanglement breaking, we apply it to a part of a maximally entangled state of two qubits $|\psi\rangle_+$. The resulting state is

$$K_0^{(\text{ph})}\,|\psi_+\rangle\langle\psi_+|\,K_0^{(\text{ph})\dagger} + K_1^{(\text{ph})}\,|\psi_+\rangle\langle\psi_+|\,K_1^{(\text{ph})\dagger} = \left(1-\frac{\delta_{\text{ph}}}{2}\right)|\psi_+\rangle\langle\psi_+| + \frac{\delta_{\text{ph}}}{2}\,|\psi_-\rangle\langle\psi_-|\,. \tag{2.1.9}$$

This is a Bell diagonal state, and as we have shown, it is separable if all the eigenvalues are smaller or equal than $\frac{1}{2}$. This is only the case if $\delta_{\text{ph}} = 1$.

Depolarising channel

The depolarising channel describes a process in which a qubit may be affected by the three Pauli matrices with equal probability $\frac{\delta_{\text{pol}}}{3}$. There is also a probability $1-\delta_{\text{pol}}$ that the qubit crosses the channel being unchanged. This models decoherence processes due to an interaction with a large environment that does not have a preferred direction.

The four Kraus operators are

$$K_0^{(\text{pol})} = \sqrt{1-\delta_{\text{pol}}}\sigma_0, \qquad K_{x,y,z}^{(\text{pol})} = \sqrt{\frac{\delta_{\text{pol}}}{3}}\,\sigma_{x,y,z}. \tag{2.1.10}$$

This channel is also unital for any value of δ_{pol}. Like before, it is easy to check if the channel is entanglement breaking. Applying it to the Bell state $|\psi_+\rangle\langle\psi_+|$, we obtain a Werner state (a Bell diagonal state where three of the eigenvalues are the same):

$$\begin{aligned} &K_0^{(\text{pol})} |\psi_+\rangle\langle\psi_+| K_0^{(\text{pol})\dagger} + \sum_{n=1}^{3} K_n^{(\text{pol})} |\psi_+\rangle\langle\psi_+| K_n^{(\text{pol})\dagger} = \\ &(1-\delta_{\text{pol}}) |\psi_+\rangle\langle\psi_+| + \delta_{\text{pol}} \frac{\mathbb{I}}{4}. \end{aligned} \tag{2.1.11}$$

This state is separable, i.e. the depolarizing channel is EBC, for $\delta_{\text{pol}} \in [1/2, 1]$.

Amplitude damping channel

This channel describes the process in which some of the energy is lost, i.e. a qubit in the upper state $|1\rangle$ has a probability δ_{ad} of decaying to the "ground state" $|0\rangle$. The Kraus operators that describe this process are

$$\begin{aligned} K_1^{(\text{ad})} &= |0\rangle\langle 0| + \sqrt{1-\delta_{\text{ad}}}|1\rangle\langle 1|, \\ K_2^{(\text{ad})} &= \sqrt{\delta_{\text{ad}}}|0\rangle\langle 1|, \end{aligned} \tag{2.1.12}$$

with $0 \leq \delta_{\text{ad}} \leq 1$. This channel is nonunital. We have

$$\begin{aligned} K_0^{(\text{ad})} K_0^{(\text{ad})\dagger} + K_1^{(\text{ad})} K_1^{(\text{ad})\dagger} &= |0\rangle\langle 0| + (1-\delta_{\text{ad}}) |1\rangle\langle 1| + \delta_{\text{ad}} |0\rangle\langle 0| \\ &= (1+\delta_{\text{ad}}) |0\rangle\langle 0| + (1-\delta_{\text{ad}}) |1\rangle\langle 1| \end{aligned} \tag{2.1.13}$$

The amplitude damping channel is entanglement breaking for $\delta_{\text{ad}} = 1$. To see this, we apply it to a qubit in a Bell state, $|\psi_+\rangle\langle\psi_+|$.

$$\begin{aligned} &K_0^{(\text{ad})} |\psi_+\rangle\langle\psi_+| K_0^{(\text{ad})\dagger} + K_1^{(\text{ad})} |\psi_+\rangle\langle\psi_+| K_1^{(\text{ad})\dagger} = \\ &\tfrac{1}{2} |10\rangle\langle 10| + \frac{1-\delta_{\text{ad}}}{2} |01\rangle\langle 01| + \frac{\sqrt{1-\delta_{\text{ad}}}}{2} (|01\rangle\langle 10| + |10\rangle\langle 01|) + \frac{\delta_{\text{ad}}}{2} |00\rangle\langle 00| \end{aligned} \tag{2.1.14}$$

In the standard basis, the partial transpose of this matrix has the form:

$$(\mathbb{I}\otimes\Phi)\left|\psi_+\right\rangle\!\left\langle\psi_+\right|^{T_Y} = \begin{pmatrix} \frac{\delta_{\rm ad}}{2} & 0 & 0 & \sqrt{1-\delta_{\rm ad}} \\ 0 & \frac{1-\delta_{\rm ad}}{2} & 0 & 0 \\ 0 & 0 & \frac{1}{2} & 0 \\ \sqrt{1-\delta_{\rm ad}} & 0 & 0 & 0 \end{pmatrix}. \tag{2.1.15}$$

For $\delta_{\rm ad} \neq 1$, this matrix has one negative eigenvalue $\frac{1}{4}\left(\delta_{\rm ad} - \sqrt{\delta_{\rm ad}^2 - 16\delta_{\rm ad} + 16}\right)$. We conclude that the amplitude damping channel is entanglement breaking only if $\delta_{\rm ad} = 1$.

2.2 Strategies for quantum communication

We are ready to introduce three classic communication protocols that take advantage of pre-shared long-distance entanglement. The applications of long-distance entanglement are very broad, but these three protocols are very simple and effective, and they are enough to motivate the content of this thesis.

2.2.1 Quantum dense coding

We mentioned in sec. 1.4 that the Holevo bound limits the amount of information that can be extracted from N qubits to be at most N classical bits [43]. However, if a qubit belongs to an entangled state it can carry the information of a pair of classical bits. This does not violate the Holevo theorem because information is actually stored in the larger entangled system. This strategy is known as quantum dense coding or superdense coding.

Suppose that Alice and Bob are sharing a maximally entangled state, i.e. Alice is holding a qubit A and Bob a qubit B in their respective distant labs, and the state of the system is $\left|\phi_+\right\rangle_{AB} = \frac{1}{\sqrt{2}}(|00\rangle + |11\rangle)$. Alice can perform one of four unitary operations, described by the Pauli matrices, on her qubit A:

1. $\sigma_0 \left|\phi_+\right\rangle_{AB} = \left|\phi_+\right\rangle_{AB}$;

2. $\sigma_x |\phi_+\rangle_{AB} = |\psi_+\rangle_{AB}$;

3. $-i\sigma_y |\phi_+\rangle_{AB} = |\psi_-\rangle_{AB}$;

4. $\sigma_z |\phi_+\rangle_{AB} = |\phi_-\rangle_{AB}$,

in this way encoding her information in four orthogonal states of the two qubits. When she sends qubit A to Bob, provided that there is no noise in the quantum channel, he will be able to perform a measurement on the Bell basis of A and B, and decode the two bits of Alice's message without loss.

Protocols for dense coding with less entangled states are also possible [61, 62], even deterministically [63].

2.2.2 Quantum key distribution

The secret of a safe cryptography is a good key distribution. Once Alice and Bob have the same key, a random succession of bits, or another alphabet, they can use it to encode and decode any message. To encode the message, Alice sums (modulo two), the key to the message, and to decode it, Bob sums the same key to the encoded message. If the key is completely random and as long as the message, it is impossible to decipher the encrypted message without the key.

The first idea for entanglement-based quantum cryptography [64] was to use entangled states to safely distribute a random key. If Alice and Bob share a Bell state $|\phi_+\rangle_{AB}$, they can perform local measurements, on the observables σ_y or σ_x and, if the measurements are in the same basis, that were publicly shared in a classical channel, the measurement outcomes, $+1$ or -1, will be perfectly correlated and can be used to read the key. The key does not exist before the measurement and any observation that a spy could make on the state could be detected by the legitimate parties.

2.2.3 Quantum teleportation

Suppose that Alice wants to send Bob a particle. One way to do it is to send the particle physically through a quantum channel.

However, suppose that such a channel is not available at the moment. They could just communicate, over the classical channel, the result of Alice's measurement on the particle, so that Bob can try to rebuild it in his lab. But the fidelity with the original state, on average, would be less than 100%. A much better way is to use their pre-shared entanglement (if they have any) to perform a quantum teleportation protocol, transferring the state of one of her particles into one of Bob's by using only classical communication. Of course the initial entanglement must have been originally distributed through a quantum channel, that is no longer available.

A teleportation protocol goes as follows [65]. Alice and Bob share initially a maximally entangled state $|\phi_+\rangle$ between two of their qubits, A and B. In her lab, Alice holds another qubit, C, in a generic unknown state $|q\rangle = \alpha\,|0\rangle + \beta\,|1\rangle$, that she wishes to transmit to Bob. So, the total initial state that they share is

$$|\phi\rangle_{ABC} = |\phi_+\rangle_{AB} \otimes |q\rangle_C = \frac{1}{\sqrt{2}}\left(|00\rangle + |11\rangle\right)_{AB} \otimes \left(\alpha\,|0\rangle + \beta\,|1\rangle\right)_C . \tag{2.2.1}$$

With some manipulation, we can rewrite it as

$$\begin{aligned} |\phi\rangle_{ABC} = \frac{1}{2}\Big[\ & |\phi_+\rangle_{AC} \otimes (\alpha|0\rangle + \beta|1\rangle)_B \ + \ |\phi_-\rangle_{AC} \otimes (\alpha|0\rangle - \beta|1\rangle)_B \ + \\ & |\psi_+\rangle_{AC} \otimes (\beta|0\rangle + \alpha|1\rangle)_B \ + \ |\psi_-\rangle_{AC} \otimes (\beta|0\rangle - \alpha|1\rangle)_B\Big]. \end{aligned} \tag{2.2.2}$$

Alice performs locally in her lab a Bell measurement (a projective measurement onto the Bell basis) over A and C, projecting the state into one of the terms in the above sum. At this point, she tells the result to Bob via the classical channel that they share. Depending on the measurement result, Bob applies a unitary Pauli operation on B. If the outcome of the measurement was $|\phi_+\rangle$, he applies the operation σ_0 that leaves the state identical. If the result was $|\phi_-\rangle$ however, he needs to apply the unitary $-i\sigma_y$. If $|\psi\rangle_+$ he applies σ_x and finally if Alice measured $|\psi_-\rangle$, he applies the matrix σ_z. At the end of this process, A and C are sharing a maximally entangled state in Alice's lab, while B is perfectly uncorrelated to them and in the state that C had initially. Note that the knowledge of $|q\rangle_C$ is not required at any point of this process.

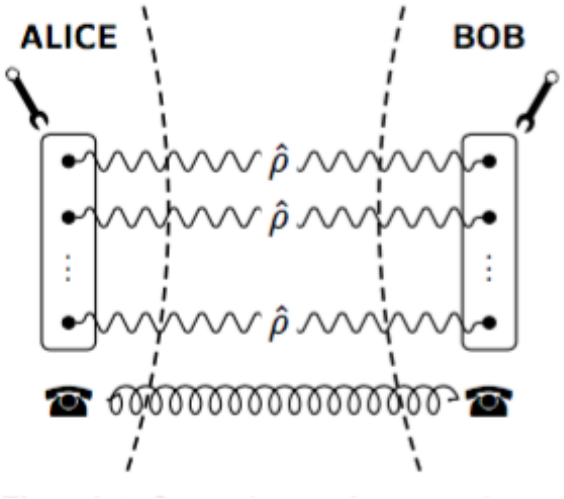

Figure 2.2: Figure taken from [66]. General entanglement distillation protocol. Alice and Bob initially share many pairs of entangled systems, and they are only allowed to perform local quantum operations and communicate with each other classically.

Alice and Bob have successfully teleported the state of C to Bob's lab. Teleportation can be realized with non-maximally entangled states [37], although the fidelity of the final state of B to the initial state of C in general is not 100%.

2.3 Manipulating long-distance entanglement

In this section, we will describe some protocols and techniques that can be used to distill, create, or increase long-distance entanglement.

2.3.1 Entanglement distillation

In the three protocols of quantum communication that we have discussed, a pre-shared, maximally entangled state was used. Similar protocols would also work with a smaller amount of entanglement but they may be less effective, or less simple to implement. Sometimes, then, it is better to have fewer copies of a maximally entangled state than many of a less-than-maximally entangled state.

The name *entanglement distillation* refers to all the protocols in which n initial copies of a bipartite entangled quantum system are transformed into $m < n$ copies

of maximally entangled (or, in general, more entangled) states without using a quantum channel, i.e. only by LOCC operations. This is depicted in Fig. 2.2.

As an example, we describe a distillation protocol introduced in [67,68] and thoroughly described in [66], that is a revisitation of one of the first proposed distillation protocols [69].

Suppose that Alice and Bob have each n qubits that form n copies of the Bell diagonal state $\rho^(0)_{pair} = A\,|\phi_+\rangle\langle\phi_+| + B\,|\psi_-\rangle\langle\psi_-| + C\,|\psi_+\rangle\langle\psi_+| + D\,|\phi_-\rangle\langle\phi_-|$, with $A \geq B \geq C \geq D$. Recall from sec. 1.3.2 that this state is only entangled if $A > 0.5$. They take two pairs of entangled qubits (pair 1 and pair 2) and perform a protocol in four steps:

Step 1: Local rotations

Alice performs the rotations $U_A = \frac{1}{\sqrt{2}}(\mathbb{I} - \sigma_X)$ on both her qubits A_1 and A_2 individually. Bob rotates his qubits with the unitary $U_B = \frac{1}{\sqrt{2}}(\mathbb{I} + \sigma_X)$. At this point, the state of the two pairs is $\rho^{(1)}_{pair} = A\,|\phi_+\rangle\langle\phi_+| + D\,|\psi_-\rangle\langle\psi_-| + C\,|\psi_+\rangle\langle\psi_+| + B\,|\phi_-\rangle\langle\phi_-|$.

Step 2: CNOT gate

Now, Alice an Bob apply a CNOT gate on their qubits, using the first one as a control. The action of this gate in the standard basis is $|0a\rangle \to |0a\rangle$, $|1a\rangle \to |1\rangle\,\sigma_x\,|a\rangle$. In other words, if the control qubit is in $|0\rangle$, it does nothing to the target qubit, if the control is in $|1\rangle$, the gate flips the state of the target qubit. At this point, the state is written as

$$\begin{aligned}
\rho^{(2)}_{12} = &\Big(A^2\,|\phi_+\rangle\langle\phi_+|\,CD\,|\psi_-\rangle\langle\psi_-| + C^2\,|\psi_+\rangle\langle\psi_+| + AB\,|\phi_-\rangle\langle\phi_-|\,\Big)_1 \otimes |\phi_+\rangle\langle\phi_+|_2 \\
&+ \Big(B^2\,|\phi_+\rangle\langle\phi_+|\,CD\,|\psi_-\rangle\langle\psi_-| + D^2\,|\psi_+\rangle\langle\psi_+| + AB\,|\phi_-\rangle\langle\phi_-|\,\Big)_1 \otimes |\phi_-\rangle\langle\phi_-|_2 \\
&+ \Big(BD\,|\phi_+\rangle\langle\phi_+|\,CB\,|\psi_-\rangle\langle\psi_-| + BD\,|\psi_+\rangle\langle\psi_+| + AD\,|\phi_-\rangle\langle\phi_-|\,\Big)_1 \otimes |\psi_-\rangle\langle\psi_-|_2 \\
&+ \Big(AC\,|\phi_+\rangle\langle\phi_+|\,AD\,|\psi_-\rangle\langle\psi_-| + AC\,|\psi_+\rangle\langle\psi_+| + BC\,|\phi_-\rangle\langle\phi_-|\,\Big)_1 \otimes |\psi_+\rangle\langle\psi_+|_2\,.
\end{aligned} \tag{2.3.1}$$

Step 3: measurement

Now Alice and Bob perform a measurement on the basis of σ_z on their second pair of qubits.

Step 4: phone call and post-selection

Alice and Bob communicate the measurement result to each other. If they obtained the same result (00 or 11), they keep the first pair of qubits (success) and discard the second pair, otherwise they discard everything.

If the protocol was successful, the state of the first pair is now

$$\begin{aligned}\rho_{pair1}^{(s)} &= \frac{1}{p_s}\left(\langle 00|_2\,\rho_{12}^{(2)}\,|00\rangle_2 + \langle 11|_2\,\rho_{12}^{(2)}\,|11\rangle_2\right)\\ &= \frac{1}{p_s}\Big[(A^2+B^2)\,|\phi_+\rangle\langle\phi_+| + 2CD\,|\psi_-\rangle\langle\psi_-|\\ &+ (C^2+D^2)\,|\psi_+\rangle\langle\psi_+| + 2AB\,|\phi_-\rangle\langle\phi_-|\,\Big],\end{aligned} \tag{2.3.2}$$

with probability of success $p_s = (A+B)^2 + (C+D)^2$. Notice that, for the discarded state, $\rho_1^{(u)} = \frac{1}{1-p_s}\left(\langle 01|_2\,\rho_{12}^{(2)}\,|01\rangle_2 + \langle 10|_2\,\rho_{12}^{(2)}\,|10\rangle_2\right)$, the largest eigenvalue is always smaller than A, so that state is not getting closer to a maximally entangled one. This protocol maps Bell diagonal states into Bell diagonal states, with coefficients of the form

$$\begin{pmatrix} A\\ B\\ C\\ D\end{pmatrix} \to \frac{1}{p_s}\begin{pmatrix} A^2+B^2\\ 2CD\\ C^2+D^2\\ 2AB\end{pmatrix} \tag{2.3.3}$$

and it is successful, i.e. $\langle\phi_+|\,\rho_{pair1}^{(s)}\,|\phi_+\rangle > \langle\phi_+|\,\rho_{pair}^{(0)}\,|\phi_+\rangle$, as long as $A > 0.5$ and $B < \sqrt{A} - A$. So, Alice and Bob may chose to start again and apply the same protocol on the saved pair and another pair prepared in the same way. It was shown in [67] and proven in [68] that, for a Werner state, i.e. if $B = C = D = (1-A)/3$, this protocol alway converges to a Bell state after a sufficient number of iterations. Notice that the Werner state always satisfies $B < \sqrt{A} - A$ for $A \geq 0$.

Of course this protocol is quite wasteful, since at least half of the qubits are discarded. A full theory of entanglement distillation does not exist yet, and is not known which is the optimal protocol for a given state.

It was shown [70] that a similar protocol to the one above is effective for any two-qubit entangled states. The PPT entangled states are impossible to distill [71]

and for this reason, it is often said that they have a "bound" entanglement.
Other classes of states display a different kind of bound entanglement [72], in the sense that entanglement distillation is not feasible, although they are not separable. These are states of tripartite systems composed of particles A, B and C, that are entangled in one bipartition, such as $A : BC$, but separable in the other two $B : AC$ and $C : AB$. Such entanglement cannot be distilled between any pair of particles. These tripartite bound entangled states play a crucial role in entanglement distribution, particularly with separable states [73].

2.3.2 Entanglement swapping

Entanglement swapping [74, 75] is a protocol for entanglement distribution in which two particles in an entangled state are teleported to a different place. A simple example is given in [76]. Say that Alice has in her lab two maximally entangled qubits, A and C, and Bob has another maximally entangled pair B and D:

$$|\psi_{ABCD}\rangle = |\phi_+\rangle_{AC} \otimes |\phi_+\rangle_{BD} \,. \tag{2.3.4}$$

This state can be rewritten also as

$$|\psi_{ABCD}\rangle = \frac{1}{2}\Big[\, |\phi_+\rangle_{AB}\otimes|\phi_+\rangle_{DC}+|\phi_-\rangle_{AB}\otimes|\phi_-\rangle_{DC}+|\psi_+\rangle_{AB}\otimes|\psi_+\rangle_{DC}+|\psi_-\rangle_{AB}\otimes|\psi_-\rangle_{DC} \,\Big]. \tag{2.3.5}$$

At this point, Alice sends particle C to Bob, through a noiseless quantum channel. When Bob receives the particle, he performs a Bell measurement on D and C. Particles A and B are now maximally entangled. This protocol does not offer an advantage compared to simply sending C through the quantum channel, at least in the absence of noise. However, it is interesting to see that systems that have never interacted directly with each other can get entangled. This is the basic principle of what we will call "indirect distribution" later in the thesis.

2.3.3 Entanglement distribution with separable states

In the entanglement swapping protocol illustrated above, one qubit that is maximally entangled crosses the channel that separates Alice from Bob. In this sense, this protocol is not very different from the one where B and A interact directly in Alice's lab before B is sent to Bob: in both cases a maximally entangled particle goes through the channel and, in the end, a maximally entangled state is shared between the labs.

In 2003, Cubitt *et al.* [73] showed that it is possible to entangle particles that do not interact directly with each other, if they exchange a separable particle. The protocols that respond to this property are often called $EDSS$ (entanglement distribution with separable states) protocols. We briefly illustrate one of the protocols proposed by Cubitt *et al.*, as illustrated in fig. 2.3.

Step 1: Initially Alice holds two qubits, A and C, in her lab. The state at this point is

$$\rho^{(1)}_{ABC} = \frac{1}{6}\sum_{k=0}^{3} |\psi_k\rangle\langle\psi_k|_A \otimes |\psi_{-k}\rangle\langle\psi_{-k}|_B \otimes |0\rangle\langle 0|_C + \frac{1}{6}\sum_{k=0}^{1} |ij\rangle\langle ij|_{AB} \otimes |1\rangle\langle 1|_C\,, \quad (2.3.6)$$

where $|\psi_k\rangle = \frac{1}{\sqrt{2}}\left(|0\rangle + e^{ik\pi/2}\,|1\rangle\right)$. This state is separable in all bipartitions, being explicitly written in a separable form.

Step 2: Now Alice performs a CNOT operation on two qubits, A and C, in her lab. The state becomes

$$\rho^{(2)}_{ABC} = \frac{1}{3}\,|GHZ\rangle\langle GHZ| + \frac{1}{6}\Big(\,|001\rangle\langle 001| + |010\rangle\langle 010| + |101\rangle\langle 101| + |110\rangle\langle 110|\,\Big). \quad (2.3.7)$$

This state is still separable in the partition $C : AB$. This can be seen from the fact that the state is symmetrical under the exchange of qubits B and C, and the operation cannot have affected the entanglement $B : AC$, that was initially zero. Particle C, then is still separable from the rest of the system.
However, this state is entangled in the partition $A : BC$, and this is what makes the distribution effective. The state is an example of a tripartite bound-entangled

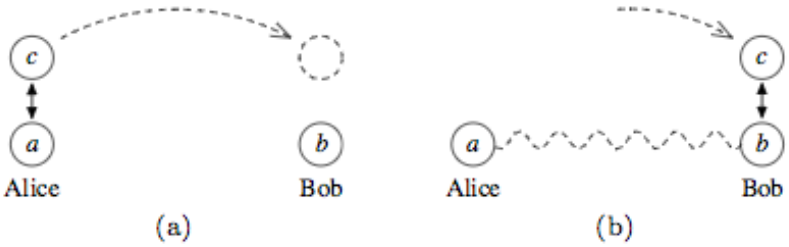

Figure 2.3: From [73]. Illustration of a discrete interaction protocol for entanglement distribution. In (a), Alice makes particles A and C interact before sending C to Bob. In (b), Bob performs an operation on B and C to localize the entanglement on A and B. The carrier C does not need to be entangled at any time with A and B.

state, and this protocol is therefore one of the few known application of bound entanglement.

Step 3: Particle C is sent to Bob, who performs another CNOT operation on B and C. The state is

$$\rho^{(3)}_{ABC} = \frac{1}{3} |\phi_+\rangle\langle\phi_+|_{AB} \otimes |0\rangle\langle 0| + \frac{2}{3}\mathbb{I}_{AB} \otimes |1\rangle\langle 1|_C \,. \tag{2.3.8}$$

At this point, Bob can measure the state of C in the standard basis to get a Bell state of A and B. Alternatively, he can apply a local quantum channel on B and C described by the Kraus operators $K_1 = |0\rangle\langle 0|_C$, $K_2 = |0\rangle\langle 0|_B \otimes |1\rangle\langle 1|_C$, $K_3 = |0\rangle\langle 1|_B \otimes |1\rangle\langle 1|_C$. After the application of the map on $\rho^{(3)}_{ABC}$ and tracing out C, A and B are left in the entangled state

$$\rho_{AB} = \frac{1}{3} \left(|\psi_+\rangle\langle\psi_+| + |00\rangle\langle 00| + |10\rangle\langle 10| \right). \tag{2.3.9}$$

In the same article, a scheme was also proposed for entanglement distribution where A and B both interact continuously with C, that stays separable at all times.

The possibility of distributing entanglement with a separable carrier is very surprising, and has generated a lot of attention. In the following we discuss some of the work that arose from this discovery.

Other works on entanglement distribution

From the existence of EDSS protocols, the question naturally arises on what limits the distribution of entanglement, if the entanglement that is sent can be zero. Two groups [77, 78] have shown that, in a protocol where the state is already prepared, the upper bound on the amount of gained entanglement is the carried quantum discord

$$|E_{A:BC} - E_{B:AC}| \leq D_{AB|C}, \tag{2.3.10}$$

where all are measured by relative entropy. This means that the amount of discord that crosses the channel is in a way the "cost" of sending entanglement.

In [77] the pre-shared discord between the labs was shown to be another limit on the distributed entanglement

$$|E_{A:BC} - E_{B:AC}| \leq D_{AC|B}, \tag{2.3.11}$$

showing that protocols with separable states cannot work if A and B have not previously interacted, directly or indirectly, with each other.

Ref. [77] also discussed the possibility of localizing the entanglement between the lab ($E_{A:BC}$) into the entanglement between the main parties. This is possible in Cubitt's protocol. In general, they showed that, if the entanglement $E_{A:BC}$ is not PPT, and if the dimensions of B and A satisfy $d_B \geq d_A$, there is always a unitary operation U_{BC} and a projective measurement that localizes the entanglement on A and B only, discarding the carrier C.

A. Kay [79] studied how Bell diagonal states of two qubits can be used for EDSS, using an initially uncorrelated carrier C, that interacts locally with A via a controlled-phase gate. He proved that these protocols are effective for a surprisingly large range of parameters, i.e. for any state except if the largest and the smallest eigenvalue sum up to $1/2$, i.e. $\lambda_1 + \lambda_4 = 1/2$, and the maximum increment in negativity in this process is $1/16$.

Some of the work by A. Streltsov *et al.* [80,81] also investigated the nature of the states in distribution processes. In [80], it was discussed that the presence of discord is not sufficient for the effectiveness of a distribution protocol and proved that no

EDSS protocol can work with a state of rank less than three, if A is a qubit. In [81], some analysis is made in order to unify all the properties of an effective distribution, based on the entanglement measure of interest, the noise in the channel, or the dimensions of the system. These considerations lead to conjecture that, whenever available, the best possible protocol is to send a particle in a pure entangled state (what we will call direct protocol), even through a noisy channel. Another interesting result is that any pure state with arbitrarily little entanglement can outperform, for some channel, a Bell state for a direct distribution of two qubits, if the measure of interest is logarithmic negativity. This resembles the result in [82], according to which the best pure state for direct distribution of singlet fraction is not a Bell state if the channel is nonunital.

An alternative, promising, way of distributing entanglement, that we are not considering in the rest of this thesis is by using quantum repeaters [83, 84]. In these schemes, entanglement is first distributed to close nodes, and then an entangled state is teleported from one node to the other, thus avoiding to have particles crossing long, noisy, quantum channels.

Protocols for entanglement distribution that are currently available are already quite effective, with entangled photons being distributed over optical fibers long hundreds of kilometers [85, 86], especially aimed at safe key distribution [87–89].

Finally, it is worth mentioning that Cubitt et al.?s EDSS protocol inspired the proposal of a protocol for distribution of secret classical information [90], in which all the correlations that are communicated are public. This classical protocol makes use of the concept of "bound information" [91, 92], that is an analog to quantum entanglement in classical cryptography.

Experimental realizations of EDSS The possibility of "sharing entanglement without sending it" [93], predicted by Cubitt *et al.*, was verified experimentally in 2013 by three groups. One of these realizations, with three photonic qubits, is described in Chapter 3. The other two were based on the theoretical proposals in [94] and [95]. In these experiments [96, 97], the parties A, B and C were three beams of light in separable Gaussian states, interfering with each other (A and C

first, then B and C) through beam splitters. It was proved that A and B become entangled with each other, and yet C stayed separable with them at all times.

3. Experimental entanglement distribution with separable states

I took part in the realization of one of the first experiments on the possibility of entanglement distribution via separable states. In this chapter, I describe the experiment, that was realized in the University of Queensland, Australia, and the data analysis that we performed in order to verify the success. The results presented in this chapter were published in Ref [98].

3.1 The model

The protocol by A. Kay [79] described in sec. 2.3.3 is suitable for experimental realization since both Bell diagonal states and a C-phase gate can be implemented with photonic qubits.

In this protocol, two qubits A and B are at first in Alice's and Bob's lab respectively, and they share a separable Bell diagonal state. The state we chose to implement in the Bell basis reads

$$\alpha_{AB} = \frac{1}{2}\left|\phi_+\right\rangle\!\left\langle\phi_+\right| + \frac{1}{4}\left|\phi_-\right\rangle\!\left\langle\phi_-\right| + \frac{1}{8}\left|\psi_+\right\rangle\!\left\langle\psi_+\right| + \frac{1}{8}\left|\psi_-\right\rangle\!\left\langle\psi_-\right|. \tag{3.1.1}$$

This state is separable because the largest eigenvalue is 1/2 (recall from the introduction, section 1.3.2 that Bell diagonal states are separable if and only if the largest eigenvalue is at most $\frac{1}{2}$).

Its separable decomposition can be written as

$$\alpha_{AB} = \frac{1}{4}\sum_{j=0}^{1} |z_j z_j\rangle\langle z_j z_j| + \frac{1}{8}\sum_{j=0}^{1} |x_j x_j\rangle\langle x_j x_j| + \frac{1}{8}\sum_{j=0}^{1} |y_j y_{1-j}\rangle\langle y_j y_{1-j}| \,. \tag{3.1.2}$$

Here $|z_0\rangle = |0\rangle$, $|z_1\rangle = |1\rangle$, $|x_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + |1\rangle)$, $|x_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - |1\rangle)$ and $|y_0\rangle = \frac{1}{\sqrt{2}}(|0\rangle + i\,|1\rangle)$, $|y_1\rangle = \frac{1}{\sqrt{2}}(|0\rangle - i\,|1\rangle)$ are the eigenvectors of the Pauli matrices σ_z, σ_x, σ_y.

In the meantime, qubit C is generated in Alice's lab in a mixed state, of the form

$$\alpha_C = \frac{1}{2}(\mathbb{I} - \tfrac{1}{2}\sigma_x) = \frac{1}{2}\left(|0\rangle\langle 0| + |1\rangle\langle 1|\right) + \frac{1}{4}\left(|0\rangle\langle 1| + |1\rangle\langle 0|\right), \tag{3.1.3}$$

so the initial state of the three qubits is

$$\alpha = \alpha_{AB} \otimes \alpha_C. \tag{3.1.4}$$

At this point, Alice applies a controlled-phase gate $\mathcal{P}_{AC}$ on A and C. This unitary gate can be interpreted as using C as a control, and applying the operation σ_z to A if C is in the state $|1\rangle$, otherwise acts as the identity. The action of this gate can be described as follows:

$$\begin{aligned} |00\rangle &\to |00\rangle \\ |01\rangle &\to |01\rangle \\ |10\rangle &\to |10\rangle \\ |11\rangle &\to -|11\rangle \,. \end{aligned} \tag{3.1.5}$$

Our numerical sampling shows that this is the optimal gate for this protocol of distribution. The initial state becomes

$$\beta = \mathcal{P}_{AC}\,\alpha\,\mathcal{P}_{AC}^{\dagger}. \tag{3.1.6}$$

By using the fact that $\mathcal{P}_{AC}\,|\phi_{\pm}\rangle_{AB} \otimes |1\rangle_C = |\phi_{\mp}\rangle_{AB} \otimes |1\rangle_C$ and $\mathcal{P}_{AC}\,|\psi_{\pm}\rangle_{AB} \otimes |1\rangle_C = |\psi_{\mp}\rangle_{AB} \otimes |1\rangle_C$, while if C is in the state $|0\rangle$, the state stays the same, it can be

checked that β can be rewritten as

$$\begin{aligned}\beta_{ABC} =&\alpha_{AB}\otimes|0\rangle\langle 0|_C+\bar{\alpha}_{AB}\otimes|1\rangle\langle 1|_C\\ &-\frac{1}{4}\Bigg[\left(\frac{1}{4}|\phi_+\rangle\langle\phi_-|_{AB}+\frac{1}{8}|\phi_-\rangle\langle\phi_+|_{AB}\right.\\ &\left.+\frac{1}{16}|\psi_-\rangle\langle\psi_+|_{AB}+\frac{1}{8}|\psi_+\rangle\langle\psi_-|_{AB}\right)\otimes|0\rangle\langle 1|_C+h.c.\Bigg],\end{aligned}\tag{3.1.7}$$

We know that this state is separable in the partition $B:AC$, because it comes from a unitary operation on only A and C on the state α. However, we want to be sure that C also remains separable with the rest of the system. It is easy to verify that the PPT criterion is satisfied, i.e. $\mathcal{N}_{C:AB}(\beta)=0$, where $\mathcal{N}_{C:AB}$ is the negativity after partial transposition, that was introduced in Sec. 1.3.2. However, in these dimensions, 2×4, PPT is not a sufficient criterion for separability.

We can, however, show explicitly a separable decomposition of the state above:

$$\begin{aligned}\beta_{ABC} &= \frac{3}{16}|z_0z_0\rangle\langle z_0z_0|\otimes|x_0\rangle\langle x_0|+\frac{3}{16}|z_1z_1\rangle\langle z_1z_1|\otimes|x_1\rangle\langle x_1|\\ &+ \frac{1}{8}|\phi^+\rangle\langle\phi^+|\otimes|z_0\rangle\langle z_0|+\frac{1}{8}|\phi^-\rangle\langle\phi^-|\otimes|z_1\rangle\langle z_1|\\ &+ \frac{1}{16}|z_0z_1\rangle\langle z_0z_1|\otimes|x_0\rangle\langle x_0|+\frac{1}{16}|z_1z_0\rangle\langle z_1z_0|\otimes|x_1\rangle\langle x_1|\\ &+ \frac{1}{16}|\phi^{+i}\rangle\langle\phi^{+i}|\otimes|y_1\rangle\langle y_1|+\frac{1}{16}|\phi^{-i}\rangle\langle\phi^{-i}|\otimes|y_0\rangle\langle y_0|\\ &+ \frac{1}{32}|z_0z_1\rangle\langle z_0z_1|\otimes|y_1\rangle\langle y_1|+\frac{1}{32}|z_1z_0\rangle\langle z_1z_0|\otimes|y_1\rangle\langle y_1|\\ &+ \frac{1}{32}|\psi^+\rangle\langle\psi^+|\otimes|y_0\rangle\langle y_0|+\frac{1}{32}|\psi^-\rangle\langle\psi^-|\otimes|y_0\rangle\langle y_0|,\end{aligned}\tag{3.1.8}$$

where $|\psi^{\pm i}\rangle=1/\sqrt{2}(|00\rangle\pm i\,|11\rangle)$. This proves that the carrier C stays actually separable.

On the other hand, we want the final state to be entangled in the partition $A:BC$, otherwise the distribution scheme has not been successful. Negativity under partial transposition is a sufficient condition for entanglement, in any dimension. Here we have

$$N_{A:CB}(\beta)=\frac{1}{16}>0.\tag{3.1.9}$$

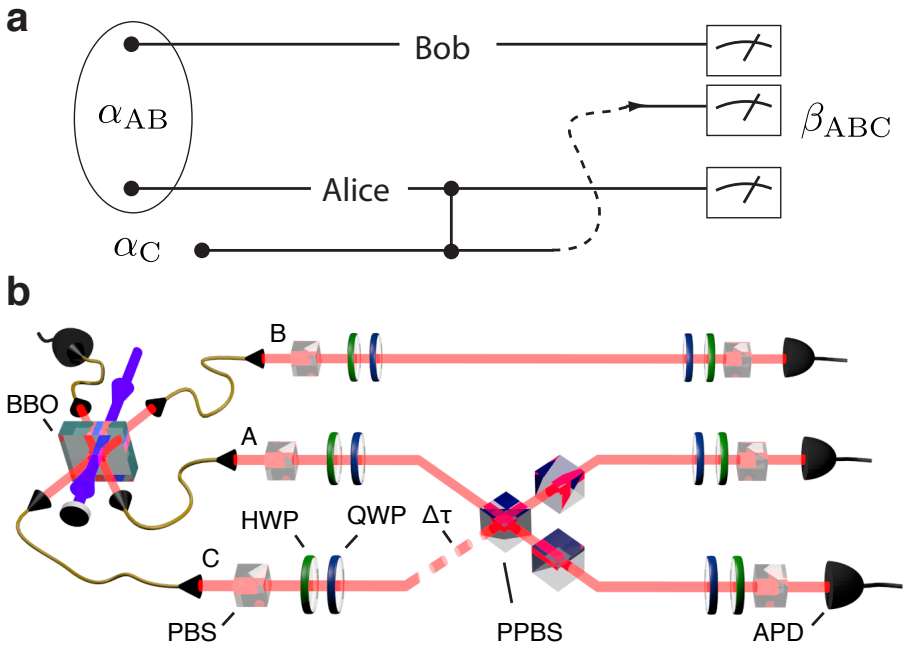

Figure 3.1: Entanglement distribution scheme. (a) A schematic representation of the protocol. (b) Experimental apparatus. Two pairs of separable photons are generated in a process of spontaneous parametric downconversion in a β-barium borate crystal (BBO), using as pump a frequency-doubled femtosecond Ti:Sapphire laser at 820 nm. Three photons are labeled as A, B and C, while the fourth serves as a trigger. The photons are initialized using a polarizing beam-splitter (PBS), a half wave plate (HWP) and a quarter wave plate (QWP). The controlled-phase gate between A and C is realized through two-photon interference at a partially polarizing beamsplitter (PPBS), $T_V = 1/3$ and $T_H = 1$ [99]. Finally, HWP, QWP and PBS, with a single-photon avalanche photodiodes (APD) and a coincidence logic operate a projective measurement on each photon.

This is enough to conclude that such protocol produces a successful distribution with a separable carrier.

3.2 The experiment

The protocol described in the previous section was implemented in Brisbane by using photonic qubits. Fig. 3.1 illustrates the experiment.

The photons are generated in two couples of separable states, via type-I parametric down-conversion in a crystal of β-barium borate (BBO). Type-I parametric

down-conversion is a process in which a nonlinear crystal (a crystal with an electric susceptibility χ that depends on the amplitude of the electric field) is stimulated by a laser (pump). The non-linear terms of the Hamiltonian of the field in the crystal couple the vacuum with a state where two photons are present. These photons are emitted simultaneously at random times, and are spatially separated and correlated in polarization.

Both these two pairs are initially in the state $|HH\rangle$, meaning that their polarizations are linear and in the horizontal direction. In our notation, this corresponds to the eigenstate of σ_z, $|z_0\rangle$. However, their initial polarization can be turned into any desired state of one qubit. For example, a half wave plate (HWP) with fast axis oriented at 45 degrees from the horizontal direction and applied on the first photon can turn its state into a vertical (V) polarization $|VH\rangle = |z_1 z_0\rangle$, while if the fast axis is oriented 22.5 degrees from the horizontal the state will turn the first photon into an anti-diagonal (A) polarization $|AH\rangle = |x_0 z_0\rangle$, and so on. A quarter wave plate, on the other hand can turn linear polarization into a circular one, and so produce the states $|y_0\rangle = |\circlearrowright\rangle$ and $|y_1\rangle = |\circlearrowleft\rangle$.

In this way, by initializing the single, unentangled, photons with wave plates, any product state of four qubits can be produced. The system composed by A and B is thus prepared in every experimental run in a pure state $|ab\rangle$ that are mixed in eq. (3.1.2), and in the same way C is prepared in the eigenbasis of its initial state, that can be written $\alpha_C = \frac{1}{4}|x_1\rangle\langle x_1| + \frac{3}{4}|x_0\rangle\langle x_0|$. Each of the settings that correspond to the preparation of the pure states are observed for a time proportional to the weight of the state in the mixture, effectively giving rise to (3.1.2). Of the four photons that are generated in the parametric downconversion, three are our A, B and C qubits, while the fourth one is used to lower the noise of the experiment, by verifying the generation of one of the photons. Photons A, B and C go then through three separate paths, thanks to optical fibers and wave guides.

Qubit B goes straight to single-photon avalanche photodiode (APD) detector, after its polarization is measured through single qubit gates. Qubits A and C interfere with each-other through a C-phase gate created, according to [99, 100], with three independent partially polarizing beam-splitters (PPBS).

Each of these beam-splitters has a different transmission and reflection amplitude for a vertically or horizontally polarized beam. The first PPBS, in which A and C interfere, has a transmission amplitude for vertical polarization $t_V^a = \sqrt{\frac{1}{3}}$, and a reflection amplitude $r_V^a = \sqrt{\frac{2}{3}}$ while, for the horizontal polarization these amplitudes are $t_H^a = 1$ and $r_H^a = 0$. The other two PPBS have $t_V^b = 1$, $r_V^b = 0$ and $t_H^b = \sqrt{\frac{1}{3}}$, $r_H^b = \sqrt{\frac{2}{3}}$. To calculate the final state for a given initial one, we multiply the initial state by these amplitudes, post-selecting the paths where a photon hits both the detectors. All the other states are discarded. We obtain

$$\begin{aligned}
|HH\rangle &\to t_H^a t_H^a t_H^b t_H^b \,|HH\rangle = \frac{1}{3}\,|HH\rangle \\
|HV\rangle &\to t_V^a t_H^a t_V^b t_H^b \,|HV\rangle = \frac{1}{3}\,|HV\rangle \\
|VH\rangle &\to t_V^a t_H^a t_V^b t_H^b \,|VH\rangle = \frac{1}{3}\,|VH\rangle \\
|VV\rangle &\to [(t_V^a t_V^a) + (ir_V^a)(ir_V^a)] t_V^b t_V^b \,|VV\rangle = -\frac{1}{3}\,|VV\rangle\,. \qquad (3.2.1)
\end{aligned}$$

By comparing this description with eq. (3.2.1), it is seen that this setup reproduces a C-phase gate with probability 1/9. Note the crucial role of the interaction between two vertically polarized photons at the first PPBS. This generates the '-' sign in the last line of eq. (3.2.1), that makes the gate entangling.

After the gate, photons A and C individually undergo the same process for a projective measurement as B. After about 387 hours and 30 000 counts in the detectors, the experimental group performed the tomography and reconstructed the probable form of the density matrix, from the numbers of photons that were detected. The experimental density matrix, as showed in Fig. 3.2, is very similar to the one that was theoretically expected. The fidelity [101] of these two matrices is $\mathcal{F}(\beta_{\rm exp}, \beta_{\rm ideal}) \equiv {\rm Tr}((\beta_{\rm exp}^{1/2}\beta_{\rm ideal}\beta_{\rm exp}^{1/2})^{1/2})^2$=0.98.

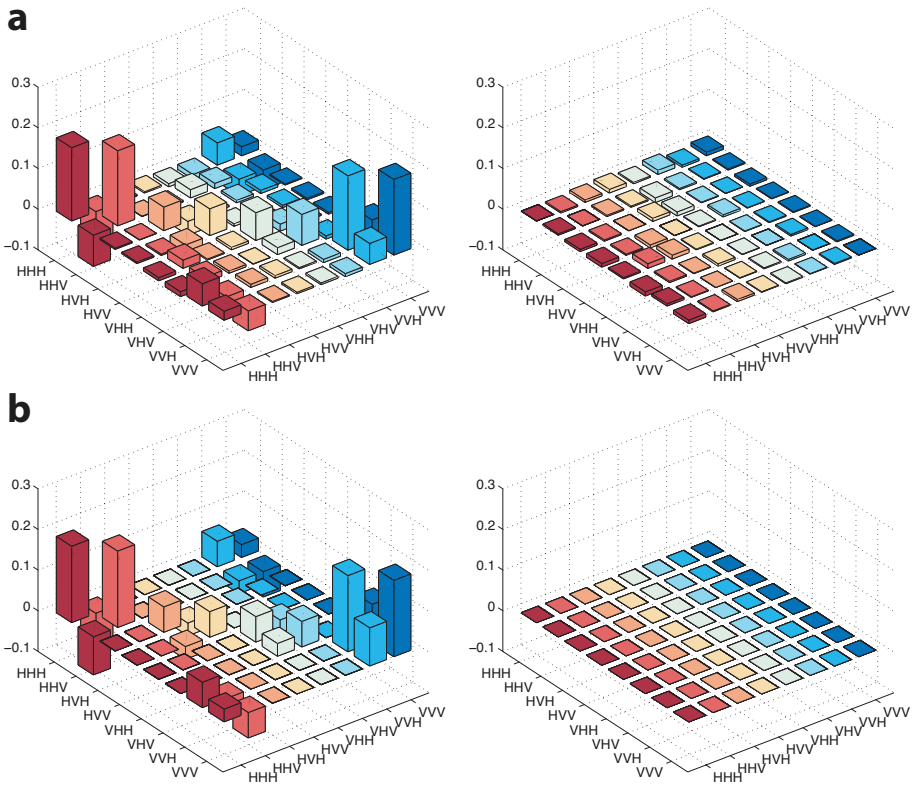

Figure 3.2: Plot of the real (on the left) and imaginary parts of the final density matrix in the ideal theoretical model (b), compared to the result of the tomography(a). The two matrices appear very similar, and the fidelity results to be of 98%.

3.3 Results and data analysis

3.3.1 Error analysis and negativity of the state

Each parametric downconversion is a random phenomenon, and is expected to occur at a random time. The probability of it happening in a given time interval is proportional to the interval length. Also, each generation of a pair of photons is independent of all the ones that preceded it. The probability that a given number of such events occurs, then, is given by the Poissonian distribution:

$$P(k \text{ generations}) = \frac{\lambda^k e^{-\lambda}}{k!}, \tag{3.3.1}$$

where λ is the average number of generations.

To evaluate the error associated with the measured state (fig. 3.2), a Monte Carlo analysis was made, building a Poissonian distribution that has the observed photon counts as average. In this way, it is possible to perform a tomography based on each of these possible photon counts, generating 10 000 density matrices that are compatible with the observed data. Calculating the fidelity between each of these 'error' density matrices, the average fidelity and its error is obtained: $\mathcal{F}_{\text{est}}$=0.967±0.007, that is still very high, although different from the one measured through the photon counts. This highlights a problem due to the finite number of observations that can be in practice made in an experiment.

It is also possible to calculate the spectrum of each of these "error states" after the partial transpose. The gained negativity is still significant $N_{A:BC} \sim 0.052$, although smaller than the one that was theoretically expected ($1/16 = 0.0625$). Unfortunately, some of the error density matrices had a negative partial transpose in the cuts $C : BA$ and $B : AC$, that means that possibly the protocol was not made a separable carrier. To overcome this problem, we mixed the initial state with a certain amount of white noise:

$$\tilde{\alpha}_{ABC} = (1-p)\alpha_{ABC} + \frac{p}{8}\mathbb{I}, \tag{3.3.2}$$

the partial transpose of the resulting state is less negative, so that for $p = 0.1667$, the error bars associated with $N_{C:AB}$ are entire in the positive quadrant, as is shown in Fig. 3.3 (a). However, this mixture has also made the protocol less 'successful' in distributing entanglement, since $N_{A:BC} = 0.0172$, as compared to the theoretical 0.0625 for the state without noise. Still, the final entanglement is non-zero and the distribution is successful. Notice that the lines traced from the expected statistics resulting from infinitely many counts (corresponding to the theoretical model) lie outside the error bars. Fig. 3.3 presents the results for many values of p.

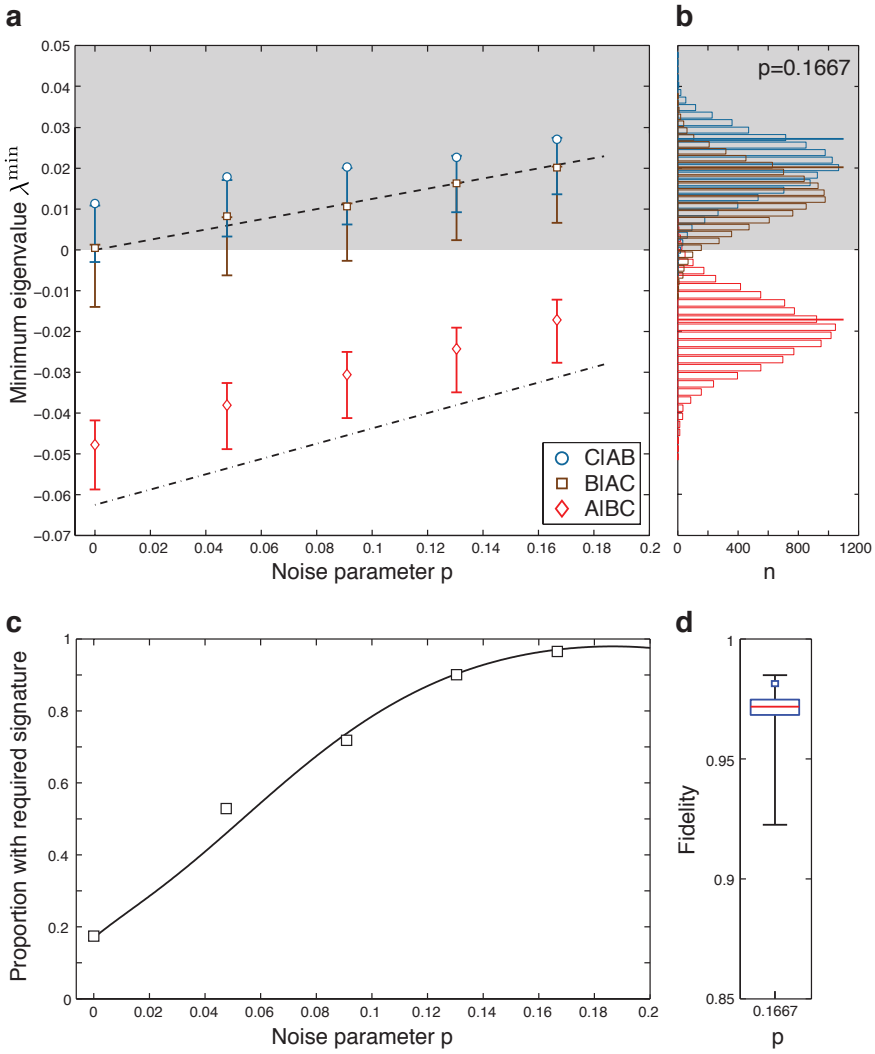

Figure 3.3: (a) Minimum eigenvalue after partial transposition $\lambda^{\min}$ for each bipartition of β_{ABC}, against white-noise admixture p, as given by eq. (3.3.2). The dashed lines show the expected values of $\lambda^{\min}$ in the $A:BC$ ($C:AB$ and $B:AC$) bipartitions if infinitely many counts were performed. The error bars are evaluated as one standard deviation from the average value in the distribution shown in (b). (b) $\lambda^{\min}$ for $p = 0.1667$. The solid line is the $\lambda^{\min}$ of the experimental state while the histogram represents the distribution of $\lambda^{\min}$ of the $10,000$ tomographic reconstructions with Poissonian distribution around the measured counts (noise matrices).(c) Proportional number of noise density matrices that have the desired positivity under partial transposition i.e. $N_{C:AB} = 0$, $N_{B:AC} = 0$ and $N_{B:AC} > 0$. The solid line is obtained by computationally adding white noise to the experimental matrix with $p = 0$. (d) Fidelity of the error matrices, represented by a box-and-whisker plot, and the theoretical state obtained by mixing white noise with parameter $p = 0.1667$. The data point is the fidelity of the experimental state to the ideal one.

3.3.2 Separability of the carrier

Even though we know that the experimental matrix, mixed with $p = 0.1667$ white noise, does not have NPT entanglement in the partition $C : AB$, we still need to be sure that the state does not have any bound entanglement in such partition, since the PPT condition is not sufficient for separability in these dimensions. For the ideal case, it can be checked that the state is separable, since eq. (3.1.7) can be rewritten in the explicitly separable form (3.1.8). For the experimental states given by the error Monte Carlo estimation, verifying the separability is more complicated, since the error density matrices did not have a strong symmetry like the ideal one. However, thanks to some similarity with the theoretical model of all the matrices, we were able to find the separable decompositions of all the noise matrices. This was achieved as follows:

1. We randomly extract pure states $|\pi_j\rangle\langle\pi_j|$, separable in the desired cut, and add them to the theoretical set of product states in Eq. (3.1.8).

2. We write the general definition of a separable state as

$$\rho_{AB|C} = \sum_j p_j \, |\pi_j\rangle\langle\pi_j| \, , \tag{3.3.3}$$

   where the index j runs through all the states of the set described in the step 1.

3. We try to recover the probabilities p_j by equating the above definition of separable state to the density matrices generated by the Monte Carlo algorithm and trying to solve analytically or numerically such equation on Mathematica.

This method worked well with only about 3000 random product pure states for each error matrix, so we were able to confirm that all the experimental states were separable in the $C : AB$ cut. For the $A : BC$ cut the negativity of the partial transposition is sufficient to ensure the presence of entanglement. In the remaining cut $B : AC$ the state should also be separable, as we started with separable state in this bipartition and C-phase was acting locally with respect to it. However, we still

verified the separability by the same method, confirming the absence of any form of entanglement in the noise.

3.3.3 Discord bound

We checked whether the discord bound (2.3.10) was saturated by our experimental states and by the noise matrices. Fig. 3.4 shows that this is not the case. A way to numerically evaluate the realtive entropy of discord in the partition $C : AB$ is

$$D_{AB|C}(\beta) = \min_{\Pi_C}[S(\Pi_C(\beta))] - S(\beta), \tag{3.3.4}$$

where S is the von Neumann entropy and Π_C denotes all the possible projective measurements over qubit C, i.e. $\Pi_C(\beta) = \Pi_0\beta\Pi_0 + \Pi_1\beta\Pi_1$, where the projectors Π_0 and Π_1 form any orthonormal basis of the Hilbert space of C.

To find the upper bound on the relative entropy of entanglement we use its definition

$$E_{A:CB}(\beta) = \min_{\rho_{A:CB}}[S(\beta|\rho_{A:CB})], \tag{3.3.5}$$

where the minimum should be taken over all separable density matrices $\rho_{A:CB}$ in the partition $A : CB$. Recall that the relative entropy is defined as $S(\beta|\rho_{A:CB}) = -\mathrm{Tr}(\beta \log \rho_{A:CB}) - S(\beta)$. The minimization is through random sampling of pure states, and we used an algorithm that, at each run, mixes the two previous separable density matrices that have the lowest relative entropy with the one we wish to check. This algorithm converges reasonably fast and allowed finding a value of the upper bound to this entanglement measure that lies below the corresponding value for the discord. Being a valid upper bound, the 'real' entanglement is verified to also be inferior to the discord. We applied the same algorithm to about 100 of the error matrices of the states in the Monte Carlo sampling used for the error evaluation. However, we immediately notice that the expectation values of both discord and entanglement lie outside the error bars calculated in such a way.

This again highlights a shortcoming of the statistical method used for the error evaluation that it is probably connected to the finite time used to take the measurements. A simulation of this experiment with a finite measurement time was

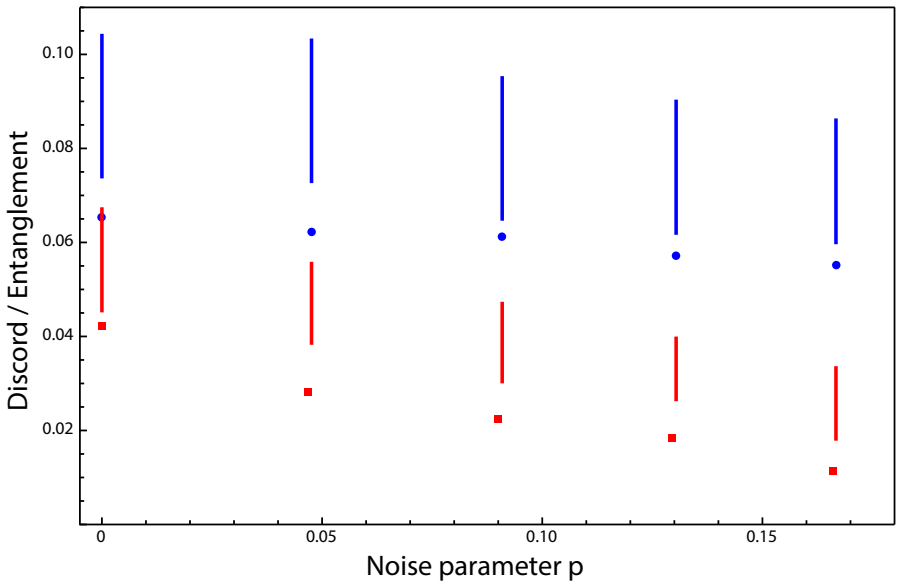

Figure 3.4: Quantum discord bounds the amount of distributed entanglement. The dots represent the relative entropy of discord $D_{AB|C}$ for the experimental density matrix. The blue lines represent the range of relative entropy of discord in which all the examined 100 error density matrices lie. The squares sign an upper bound on the relative entropy of entanglement in the final state, $E_{A:CB}$, and the red lines sign the range on which the entanglement of the error matrices lie. The fact that the experimental dots lie outside the matrices can be explained, analogously to Fig. 3.3, through the faultiness of a limited time of measurement.

carried out and the result is reported in Appendix 3 of the experimental paper [102]. An analysis on what causes statistical errors in similar experiments can be found in [103]. Since the relative entropy is a strongly non-linear function of the number of photon counts in each detector, this imperfection made in the evaluation of the states propagates, generating this abnormality in the plot.

3.4 Conclusions

An experimental realization of a EDSS protocol using photonic qubits was illustrated. In this experiment, a protocol using an initial Bell diagonal state has been reproduced. The tomography revealed a fidelity with the theoretical protocol of 98%.

The separability of the carrier and a positive increment in entanglement (measured as negativity) have been successfully verified, within the error bar, by mixing the ideal state with white noise. The analysis of the relative entropy of entanglement and of discord and the calculation of the negativity however reveal a problem with the error analysis, due to the finite measurement time.

4. Excessive entanglement distribution

In this chapter, we provide a classification of entanglement distribution protocols, based on the properties of the carrier parties. We mentioned in the introduction that entanglement distribution is possible if the parties do not interact directly with each other. Some examples are entanglement swapping, and Cubitt *et al.*'s protocol with separable carriers [73]. We will call these *indirect protocols*.

Excessive protocols are a generalization of the separable carriers protocols. In this case, we don't impose for the carrier to be separable, but only that its entanglement (in the chosen measure) is smaller than the one that is finally gained. I will next show some properties of excessive protocols. Some of them generalize properties of EDSS protocols, but others are, quite surprisingly, different.

Then, I will present a class of protocols based on a single parameter state, and discuss the transition between excessive and non-excessive protocol, depending on the particles that act as carriers. This reveals an interesting phenomenon that we will call *catalysis of excessiveness* (similarly to the catalysis of entanglement that I mentioned in sec. 1.1.1). Most results presented in this chapter are published in [104]

4.1 Direct and indirect protocols

Our first classification for an entanglement distribution protocol is illustrated in Fig.4.1. The goal of all distribution protocols is the same: Alice and Bob keep in their respective labs two subsystems, A and B that are entangled with each other. There are usually two ways this can be achieved.

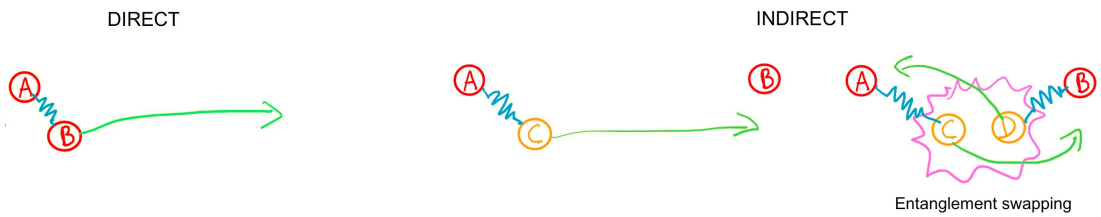

Figure 4.1: Representation of direct and indirect protocols for the distribution on entanglement. In a direct protocol, the entanglement is carried by one of the main parties that are meant to be entangled. In indirect distribution protocols, the carrier is one or many ancilla. As examples of indirect protocols, we note uni-directional distribution schemes (the model that is most explored in this thesis), and an entanglement swapping process.

We call *direct* protocols all the protocols in which the entanglement is carried by one of the main subsystems. This is the simplest, most intuitive, way of distributing entanglement. Particles A and B interact locally in Alice's lab, and then one of them, that we call B, is sent to Bob. Provided that entanglement was successfully generated by the local interaction, and that such entanglement is not entirely disrupted before B reaches Bob, Alice and Bob will eventually share two entangled particles that they can use, if they need it.

Indirect protocols are more complex, because they involve the use of ancillary systems. In these protocols, A and B are initially displaced, but their entanglement is increased by interaction with a common subsystem. Entanglement swapping, introduced in 6.3.6 is an example of indirect protocol. Another example is a unidirectional distribution protocol, like the EDSS protocol described in Sec. 2.3.3: Alice performs a local operation, in her lab, on her subsystem, A, and the carrier C. Then she sends C to Bob, who in turn performs some operations aimed at increasing the entanglement between A and B, after C has been discarded.

4.2 Excessive and non-excessive protocols

Our second classification comes from the first one, and in a way it is a quantitative formulation of it. In both direct and indirect protocols, as we have defined them, there is a carrier system, whether it is an ancilla or not. The change of entanglement

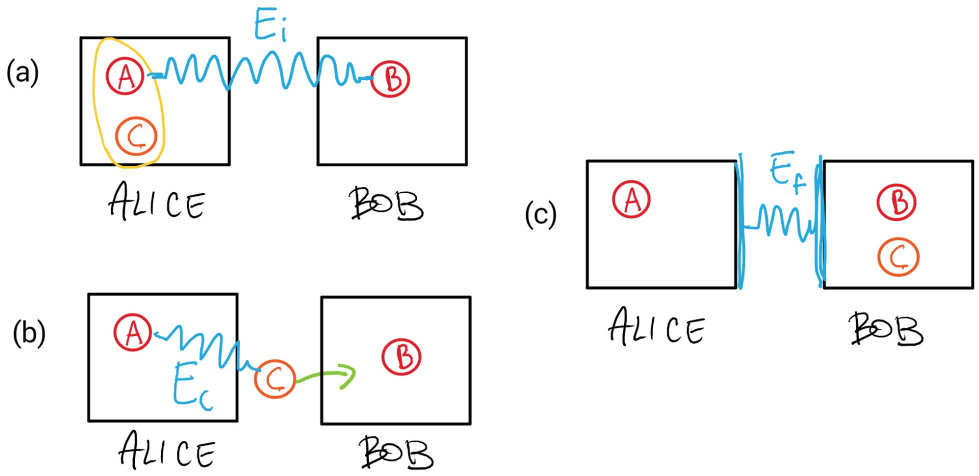

Figure 4.2: Excessive and non-excessive protocols. (a) A and B initially share some correlations, and then A and C interact locally in Alice's lab. The initial entanglement is $E_{AC:B}$. (b) C travels from Alice to Bob, carrying some entanglement that amounts to $E_{AB:C}$. (c) Now Alice and Bob share some entanglement, given by $E_{A:CB}$. If $E_{A:CB} - E_{AC:B} > E_{AB:C}$, the protocol is excessive

between Alice and Bob's lab is $\delta E \equiv E_{\rm f} - E_{\rm i}$. We call the entanglement of the carrier system, while it is traveling, $E_{\rm com}$.

We already know, from the existence of EDSS protocols, that the entanglement gain can exceed the entanglement that is communicated. This inspires our second classification:

$$\begin{aligned} \delta E &\leq E_{\rm com}: && \text{Non-excessive protocol,} \\ \delta E &> E_{\rm com}: && \text{Excessive protocol.} \end{aligned}$$

In a direct protocol, the initial shared entanglement between Alice's and Bob's laboratories is usually zero, because all the particles are at Alice's at that point. The carried entanglement is $E_{A:B}$, provided that there is no noise in the communication channel. In this case, $\delta E = E_{\rm com}$. Direct protocols are always non-excessive.

Another example is our main model of indirect protocols, unidirectional indirect communication (we will just call them indirect protocols in the following). In this case, the initial entanglement is $E_{\rm i} = E^{(i)}_{A:B}$. We assume that, at the initial stage,

and before any interaction with A or B, the ancilla C is completely uncorrelated (factorized) to the rest of the system, i.e.

$$\rho = \alpha_{AB} \otimes \alpha_C. \tag{4.2.1}$$

Consequently we must have $E^{(i)}_{A:B} = E^{(i)}_{AC:B}$. This assumption is usually verified in realistic scenarios.

On the other hand, Bob can only act on B and C once the carrier has reached his lab. In order to discard C and localize the entanglement on A and B, all he has available are local operations in his lab, including unitary operations, quantum channels or projective measurements on B and C. Then, the final entanglement $E_f = E_{A:B} \leq E_{A:BC}$ (because the partial trace is a local quantum channel). Therefore

$$E_f - E_i \leq E_{A:CB} - E_{AC:B} \equiv E_{\text{fin}} - E_{\text{in}} \equiv \Delta E. \tag{4.2.2}$$

Here, we defined ΔE as

$$\Delta E \equiv E_{\text{fin}} - E_{\text{in}}. \tag{4.2.3}$$

In summary, in our notation, E_{fin} and E_{in} represent the initial and final entanglement between the laboratories, before any localizing operation, and are in general different from what we previously called E_f and E_i, the final and initial entanglement between the main particles A and B. However, we always have

$$\delta E \leq \Delta E. \tag{4.2.4}$$

We recall the theorem in [77], mentioned in 2.3.3, according to which, as long as $\dim(A) \geq \dim(B)$, there is a local operation in Bob's lab that entirely localizes the entanglement, measured by negativity, between the labs into the main parties. With this operation, then, we obtain $\delta E = \Delta E$ as long as eq. (4.2.1) is satisfied.

Whenever we talk about indirect protocols, in the following, we will then use the

following classification, unless stated otherwise:

$$\begin{aligned} E_{A:BC} - E_{B:AC} &\leq E_{C:AB}: && \text{Non-excessive protocol,} \\ E_{A:BC} - E_{B:AC} &> E_{C:AB}: && \text{Excessive protocol.} \end{aligned}$$

.

4.2.1 Excessive distribution and entanglement measures

Since all entanglement measures are not equivalent to each other, whether a protocol is excessive or non-excessive may depend on the measure of choice. We will show that some particular choices of measures, suitable for pure states, never allow any excessive distribution to succeed. Other measures allow excessive protocols even with pure states.

Sub-additive measures

Consider all the entanglement measures that have the sub-additivity property discussed in sec. 1.1.2. Von Neumann entropy or linear entropy of the reduced density matrices of pure states are an example of these. These measures are sub-additive because the total system always has less of the quantity of attention than the sum of the two subsystems:

$$S(\rho_{BC}) \leq S(\rho_B) + S(\rho_C). \tag{4.2.5}$$

Consider, for example, S to be the Von Neumann entropy. Then if ρ_{ABC} is a tripartite pure state, $S(\rho_{BC})$ is a measure of the entanglement $E_{A:BC}$. On the other hand, $S(\rho_B) = S(\rho_{AC}) = E_{B:AC}$ and $S(\rho_C) = S(\rho_{AB}) = E_{C:AB}$. Considering that, Eq. (4.2.5) is equivalent to our definition of non-excessive protocols in Eq. (4.2.5).

This shows that excessive distribution is impossible with pure states, if the measure of interest is the Von Neumann entropy. The same argument applies to linear entropy $S(\rho) = 1 - \mathrm{Tr}(\rho^2)$, and any other measure that is sub-additive and produces a value of entanglement when evaluated on complementary subsystems. These measures are usually defined for pure states. At this point, it can look like, as well as

EDSS, excessive distribution is not possible for pure states. However, somewhat surprisingly, this is not the case for at least some entanglement measures that are not sub-additive, as we show now.

Negativity

Negativity, introduced in sec. 1.3.2, is an entanglement measure that is relatively easy to compute. Furthermore, as a measure, it is not sub-additive, so it might be suitable for excessive protocols. As we mentioned in sec. 1.3.6, this quantity has some operational meaning, for instance, it bounds the accuracy that the state allows in a teleportation protocol.

We recall that, given a state of a bipartite system ρ_{XY} the definition of negativity is

$$N_{X:Y} = \frac{\|\rho_{XY}^{PT}\| - 1}{2}, \tag{4.2.6}$$

where $||\rho_{XY}^{PT}|| = \mathrm{Tr}(\sqrt{\rho_{XY}^{PT\,2}})$ is the trace norm of the partial transpose of ρ with respect to one of the subsystems X or Y.

We prove, however, that, as long as A is a qubit, excessive distribution of negativity is impossible for pure states. We first demonstrate the lemma:

Lemma 1. *For any tripartite pure state, we have:*

$$\sqrt{\tfrac{2}{d_A(d_A-1)}} N_{A:CB} \leq N_{AC:B} + N_{AB:C}, \tag{4.2.7}$$

where d_A is the rank of the reduced state of A.

Proof. Since the state is pure, we can always write the Schmidt decomposition (Sec.

1.1.1) of it for all the three possible bipartitions:

$$\begin{aligned} |\psi\rangle &= \sum_{\alpha=1}^{d_A} \sqrt{p_\alpha}\, |\alpha\rangle_A\, |\phi_\alpha\rangle_{BC} \\ &= \sum_{\beta=1}^{d_B} \sqrt{q_\beta}\, |\beta\rangle_B\, |\chi_\beta\rangle_{AC} \\ &= \sum_{\gamma=1}^{d_C} \sqrt{r_\gamma}\, |\gamma\rangle_C\, |\xi_\gamma\rangle_{AB}. \end{aligned} \tag{4.2.8}$$

As it can be easily verified by performing the partial transpose of the density matrices, written on the Schmidt basis, the negativities can be written in terms of the Schmidt coefficients as:

$$\begin{aligned} N_{A:CB} &= \frac{1}{2} \sum_{\alpha \neq a} \sqrt{p_\alpha p_a}, \\ N_{AC:B} &= \frac{1}{2} \sum_{\beta \neq b} \sqrt{q_\beta q_b}, \\ N_{AB:C} &= \frac{1}{2} \sum_{\gamma \neq c} \sqrt{r_\gamma r_c}. \end{aligned} \tag{4.2.9}$$

Similarly, if we calculate the linear entropy of the reduced states, we see that its sub-additivity condition also can be written in terms of the Schmidt coefficients:

$$\sum_{\alpha \neq a} p_\alpha p_a \;\leq \sum_{\beta \neq b} q_\beta q_b + \sum_{\gamma \neq c} r_\gamma r_c. \tag{4.2.10}$$

The proof proceeds from geometrical considerations: the sum on the left hand side of eq. (4.2.10) is the length of the $d_A(d_A-1)-$dimensional vector that has components $\vec{v} = (\sqrt{p_1 p_2}, \sqrt{p_1 p_3}, \ldots, \sqrt{p_{d_A-1} p_{d_A}})$. Taking the scalar product of this vector with a uniform vector in the same space, $\vec{w} = (\frac{1}{2}, \frac{1}{2}, \ldots, \frac{1}{2})$, we obtain $N_{A:CB}$ from Eq. (4.2.9). The Cauchy-Schwartz inequality, applied to vectors $\vec{v}$ and $\vec{w}$, guarantees that

$$\frac{4}{d_A(d_A-1)} N_{A:CB}^2 \leq \sum_{\alpha \neq a} p_\alpha p_a. \tag{4.2.11}$$

Furthermore, by considering that negativity is a positive number, we have

$$\begin{aligned}(N_{AC:B} + N_{AB:C})^2 &\geq N_{AC:B}^2 + N_{AB:C}^2 \\ &= \frac{1}{4}\sum_{\beta\neq b}\sqrt{q_\beta q_b}\sum_{\beta\prime\neq b\prime}\sqrt{q_{\beta\prime}q_{b\prime}} \\ &+ \frac{1}{4}\sum_{\gamma\neq c}\sqrt{r_\gamma r_c}\sum_{\gamma\prime\neq c\prime}\sqrt{r_{\gamma\prime}r_{c\prime}} \\ &\geq \frac{1}{2}\sum_{\beta\neq b} q_\beta q_b + \frac{1}{2}\sum_{\gamma\neq c} r_\gamma r_c.\end{aligned} \tag{4.2.12}$$

The last inequality holds because, in the sums, we will see the combination of certain indices two times. For instance we obtain $q_\beta q_b$ both when we multiply the term $\beta = \beta\prime$ with $b = b\prime$ and also when we multiply $\beta = b\prime$ with $b = \beta\prime$. All the remaining addends are positive or zero. Combining (4.2.11) and (4.2.12) with the inequality (4.2.10), we have proven the lemma (4.2.7). □

The previous lemma has an immediate consequence in terms of excessive protocols with negativity, if A is a qubit:

Theorem 1. *For any pure tripartite state, where A is a qubit:*

$$N_{A:CB} \leq N_{AC:B} + N_{AB:C}, \tag{4.2.13}$$

.

Proof. Since A is a qubit, we must have $d_A = 2$. In this case, (4.2.7) is equivalent to the above statement. □

This shows that no excessive protocols with negativity are possible, by using only pure states.

However, if the rank of the subsystem A is at least 3, the above lemma does not say anything about whether excessive protocols exist or not. The surprising answer is that they do exist. For instance, consider the pure state:

$$|\psi\rangle = \frac{1}{\sqrt{3}}(|200\rangle + |001\rangle + |110\rangle), \tag{4.2.14}$$

for which we have $N_{AB:C} = \sqrt{2}/3 \approx 0.471$ and $N_{A:CB} - N_{AC:B} = 1 - \sqrt{2}/3 \approx 0.529$. The distribution with this initial state is thus excessive!

This result is different from the corresponding one with EDSS protocols. This class of protocols just cannot happen with pure states, no matter how entanglement is measured, because in this case, the three parties are, at all times, in a pure state of the form $|\phi_{AB}\rangle \otimes |\phi_C\rangle$. For such state, $E_{A:BC} = E_{A:B} = E_{B:AC}$, hence the entanglement between the labs cannot increase by sending C between them.

Logarithmic negativity

Logarithmic negativity,

$$L_{X:Y} = \log_2 \|\rho_{XY}^{PT}\| = \log_2(2N_{X:Y} + 1). \tag{4.2.15}$$

is an entanglement monotone that is closely related to negativity. However, it has some different properties, for instance it is not a convex function, as was mentioned in sec. 1.3.3. It also has an important operational meaning, as it is proven to be an upper-bound to the distillable entanglement in the state. We show that this measure also allows excessive distribution with pure states.

If A is a qubit however it is not possible, which is a direct consequence of the result for negativity:

Theorem 2. *For a tripartite pure state where A is a qubit the logarithmic negativity satisfies*

$$L_{A:CB} - L_{AC:B} \leq L_{AB:C}. \tag{4.2.16}$$

Proof. From the inequality (4.2.13), we multiply both sides by two and add one. We then take the logarithm:

$$\begin{aligned} \log_2(2N_{A:CB} + 1) &\leq \log_2(2N_{AC:B} + 2N_{AB:C} + 1) \\ &\leq \log_2(2N_{AC:B} + 2N_{AB:C} + 1 + 1) \\ &\leq \log_2(2N_{AC:B} + 1) + \log_2(2N_{AB:C} + 1). \end{aligned} \tag{4.2.17}$$

The last two inequalities follow, respectively, from the monotonicity and the con-

vexity of the logarithm. This, together with the definition (4.2.15), proves the theorem. □

The excessiveness of the logarithmic negativity is a stronger condition than the one of the negativity. The above proof can be repeated for any protocol where negativity displays non-excessiveness. In other words, if a protocol is excessive for logarithmic negativity, it is also excessive for negativity, but not vice versa.

For instance, the state in Eq. (4.2.14) does not violate the inequality (4.2.16) for logarithmic negativity, nor any other state we have sampled in which $d_A = 3$. Thus, if $d_A = 3$, we were not able to prove or disprove the existence of an excessive protocol with pure states.

However, we did find an example for $d_A = 4$. In the state

$$|\psi\rangle = \frac{1}{\sqrt{103}} \left(10\,|000\rangle + |110\rangle + |201\rangle + |311\rangle\right), \tag{4.2.18}$$

$L_{AB:C} \approx 0.352$, while the entanglement gain is $L_{A:CB} - L_{AC:B} \approx 0.363$.

Excessive protocols and unification of quantum correlations

The existence of examples of pure states that allow excessive distribution may have a consequence on the search for a unified measure for general quantum correlations (entanglement and discord). As we discussed in sec. 1.4.3, the known measures of bipartite discord that are comparable to bipartite entanglement have two common properties:

- the measure for discord reduces to the measure for entanglement for pure states. This must be true, since in pure states separability in a bipartition is just equivalent to classicality, because bi-separable pure states are factorized.

- discord is usually the bound for the entanglement increment, as in Eq. (1.4.3).

Both these conditions cannot be satisfied by a unified measure involving negativity or logarithmic negativity, since a state such as (4.2.14) would violate at least one of them.

We must conclude that, if a unified measure of quantum correlations involving such measures existed, it should have some different properties from relative entropy and other measures that are known so far.

4.2.2 Some excessive protocols

Apart from the excessive protocols that can be realized with the pure states we listed above, we are now ready to illustrate an interesting class of protocols, using mixed states of five qubits. These states depend on a single parameter. Depending on which partitions we consider (in the distribution perspective, which subsystems carry the entanglement), this class of states covers all different cases of distribution protocol: excessive, non-excessive, and with separable or entangled carriers. Furthermore, it allows us to highlight an interesting phenomenon, that we called catalysis of excessiveness, similarly to the entanglement catalysis that was mentioned in sec. 1.1.1. This phenomenon was already present in previous literature, but never explicitly noted.

We now describe how the states are constructed. We start from an Absolutely Maximally Entangled (AME) state of five qubits [27,105], as described in sec. 1.2.3:

$$\begin{aligned} |\psi\rangle \;=\; & \frac{1}{4}\Big(|00000\rangle + |10010\rangle + |01001\rangle + |10100\rangle \\ & +|01010\rangle - |11011\rangle - |00110\rangle - |11000\rangle \\ & +|11101\rangle - |00011\rangle - |11110\rangle - |01111\rangle \\ & +|10001\rangle - |01100\rangle - |10111\rangle + |00101\rangle\Big). \end{aligned} \tag{4.2.19}$$

Recall that AME states have the property of being maximally entangled in any partition that is considered. We apply locally two depolarizing channels (introduced in sec. 2.1.2), acting on two of the qubits (labeled 1 and 2) of the 5-qubit AME state. These channels have the following Kraus operators:

$$\begin{aligned} K_0^{(1)} &= \frac{1}{\sqrt{2}}\mathbb{I}, \quad K_i^{(1)} = \frac{1}{\sqrt{6}}\,\sigma_i, \\ K_0^{(2)} &= \sqrt{q}\mathbb{I}, \quad K_i^{(2)} = \sqrt{\frac{1-q}{3}}\,\sigma_i. \end{aligned} \tag{4.2.20}$$

Note that the channel acting on particle 2 depends on a continuos parameter, q, while the one that acts on qubit 1 is fixed.

The depolarizing channel acting on qubit 2 is entanglement breaking (i.e any subsystem that crosses it becomes separable with the rest of the system) as long as the parameter $0 \leq q \leq 0.5$, as proved in sec. 2.1.2. On the other hand, the channel acting on particle 1 is always entanglement breaking. With this construction the separability of the state is as follows:

- $E_{1:2345} = 0$ for any q;
- $E_{2:1345} = 0$ for $0 \leq q \leq 0.5$;
- $E_{12:345} = 0$ for $0 \leq q \leq 0.5$;
- The state is entangled in any other bipartition.

We are now ready to analyze how this state can allow the implementation of excessive and non-excessive protocols, by choosing different groups of particles to play the part of the carrier C and of the main parties A and B.

For instance, let us look at fig. 4.3, where we plot the logarithmic negativities for some of the partitions. Here, Alice holds three of the qubits, $A = \{2, 4, 5\}$, and Bob the separable qubit $B = \{1\}$. The remaining entangled qubit $C = \{3\}$ is used as a carrier. The initial entanglement is $E_{B:AC} = E_{1:2345} = 0$, for any q, since qubit 1 is prepared in a separable state. As the plots in fig. 4.3 show, the final entanglement $E_{A:BC} = E_{245:13}$ is always above the communicated one $E_{C:AB} = E_{3:1245}$, making the protocol excessive, except when $q = 0, \frac{1}{4}, 1$.

Now, if we exchange B with C, making particle 1 the carrier and particle 3 Bob's system B, we see from the same figure 4.3 that the protocol is still excessive, but the carrier is now separable.

A similar analysis can be performed by considering different partitions, as made in fig. 4.4. Now Alice is holding three of the qubits, one of which is the separable qubit 1. So $A = \{1, 4, 5\}$, while $B = \{2\}$ and $C = \{3\}$. The initial entanglement, then, is zero for $0 \leq q \leq 0.5$. Notice that the range of q where the protocols is

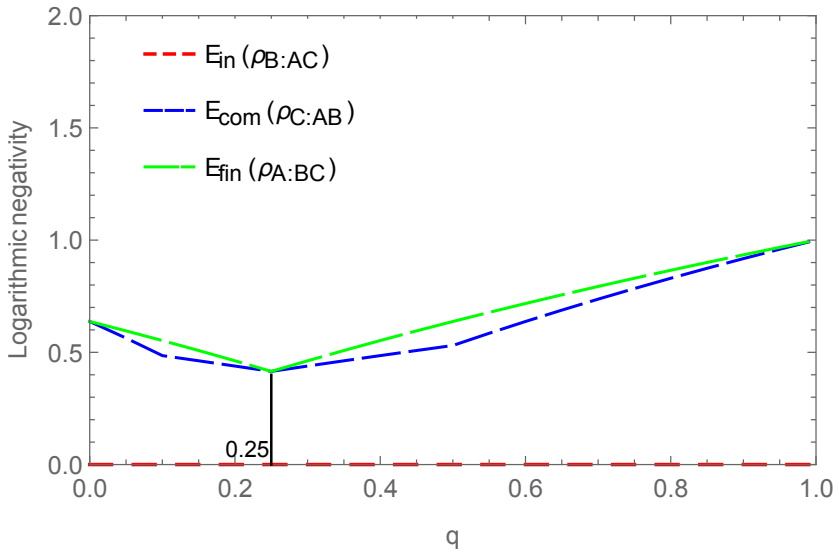

Figure 4.3: The partitions used to calculate this graph are $A = \{2, 4, 5\}$, $B = \{1\}$ and $C = \{3\}$. Since qubit 1 is separable from the rest of the system, we have $E_{\text{in}} = E_{1:2345} = 0$. The communicated entanglement, however, evaluated as logarithmic negativity, is always strictly positive and the final entanglement is always larger than the initial one, except for some special points: $q = 0, \frac{1}{4}, 1$. Everywhere else, the protocol is excessive.
By exchanging qubits 1 and 3 we obtain a different excessive protocol, with a separable carrier.

excessive is now very narrow, $0.4 \leq q \leq 0.55$, and it is partially realized with a separable initial configuration, because $E_{12:345} = 0$ for $q \in \left[0, \frac{1}{2}\right]$.

Finally, we analyze the configuration $A = \{4, 5\}$, $B = \{1, 2\}$ and $C = \{3\}$, as plotted in fig. 4.5, to expose the catalysis of excessiveness. Here the final entanglement is larger than the one plotted in fig. 4.4. Also, the interval in q in which the protocol is excessive is now broader. However, the only difference with the previous case is to have always separable qubit 1 initially with Bob. Note that this does not affect the initial entanglement. In this case, excessive protocols obtained by preliminarily sending a separable particle are more effective, even though the initial entanglement is the same. Particle 1 acts here as a catalyst.

In retrospect, the same phenomenon is present in the paper by Cubitt *et al.* [73], where the original protocol for distribution with a separable carrier is described, as well as in the protocol used for the experiment described in the previous chapter.

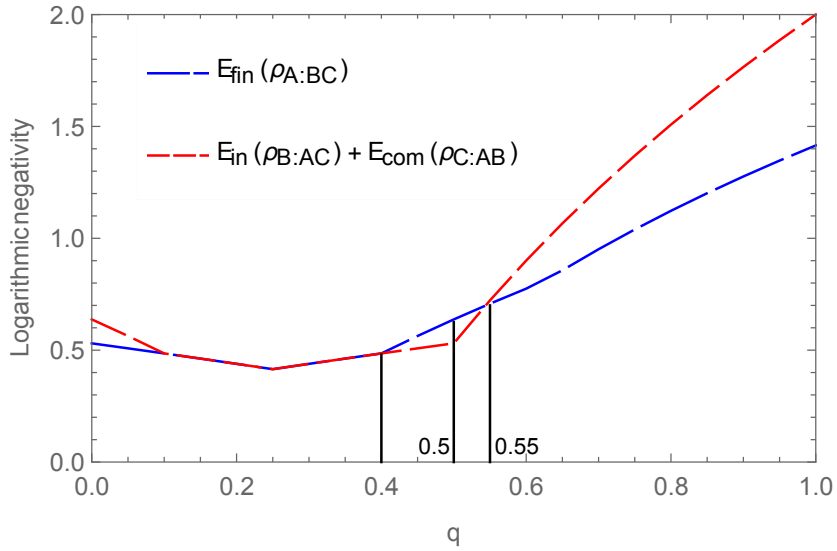

Figure 4.4: The partitions here are $A = \{1, 4, 5\}$, $B = \{2\}$ and $C = \{3\}$. For $0.4 < q \leq 0.5$ we have an excessive protocol, with separable initial state. For $0.5 < q < 0.55$ we have another excessive protocol, but now the initial, communicated and final entanglement are all nonzero. Everywhere else the protocol is non-excessive.

Consider the alternative scenario in which only the separable carrier is sent to Bob, while particle B stays with Alice the whole time. Clearly, no entanglement is gained in this way. On the other hand, if the separable particle B is initially sent, without altering the initial entanglement, we know that sending the carrier C will allow a successful excessive distribution. Particle B is then a catalyst for this excessive protocol.

4.3 Conclusions

We have characterized many classes of protocols for entanglement distribution. The simplest way to prepare an entangled state between two distant parties is by performing what we called a direct distribution protocol. However, a larger and more complex set of possibilities is offered by indirect distribution, whereby an ancilla is used to increase the entanglement between two already distant quantum systems. For instance, in indirect distribution protocols, entanglement can be sent without

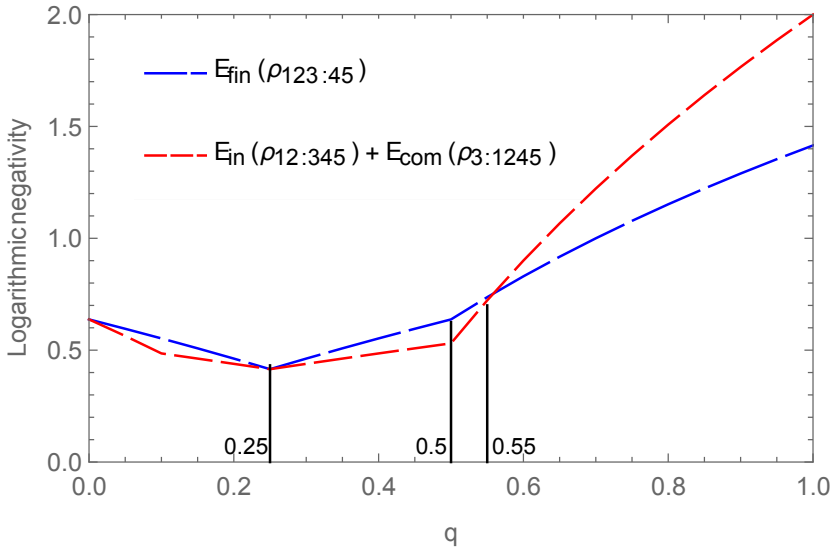

Figure 4.5: This protocol illustrates the phenomenon of catalysis of excessiveness. Having sent qubit 1 does not affect the initial entanglement, as is seen by comparing with fig. 4.4. However, it makes the entanglement gain larger, and also increases the interval in q on which the protocol is excessive.

being communicated (EDSS protocols) or, more in general, the entanglement gain can exceed the communicated entanglement (excessive protocols). While EDSS protocols are not possible with pure states, excessive protocols with an entangled carrier can be performed when the total state is pure for some entanglement measures.

A class of protocols was illustrated, showing how the entanglement gain and excessiveness (in logarithmic negativity) depend on a single parameter used in the 5-qubits state preparation. Another factor is what particles are used as carriers. We showed that separable particles can also, in some cases, act as "catalysts", making the distribution both more effective and excessive, without affecting the initial entanglement.

5. Entanglement distribution through noisy channels

In the last chapters, we have only considered a transmission through an ideal channel. However, in reality it is very difficult to avoid noise, especially if the communication has to happen through long distances. The interaction with all the other quantum particles in the environment can make a pure state mixed, or make it lose coherence and, if such particle started as entangled, some of the entanglement that was carried will probably be lost. From this perspective, indirect protocols can be helpful. Direct distribution only allows as much entanglement to reach the other location as it is communicated, thus making the loss in the channel important. On the other hand, indirect protocols allow excessiveness, i.e. beat the limit imposed by the communicated entanglement. And even if all entanglement is destroyed, and the carrier reaches the other lab being separable, it is still possible to have a significant gain, because of the existence of EDSS protocols. In particular, entanglement braking channels (EBC) do not allow any direct distribution. However, whether they allow indirect protocols, that inevitably will be EDSS is not trivial at a glance. However, we will show that this is not the case.
This work was published in **??**.

5.1 Optimal direct protocol

We now prove that, for any noisy channel and for convex entanglement monotones, the best direct protocol is made by sending qubit B through the channel if A and

B start in a pure state. Let us call our quantum channel Φ acting on B, and a quantum mixed state of two qubits, ρ_{AB}.

In the Lewenstein-Sanpera decomposition [106], any mixed state of two qubits can be written as a mixture of a separable mixed state and a pure state. We apply this to the initial density matrix ρ_{AB}:

$$\rho_{AB} = \alpha\sigma_{AB} + (1-\alpha)\left|\psi\right\rangle\left\langle\psi\right|, \tag{5.1.1}$$

where $\alpha \in [0,1]$ is a real number and σ_{AB} is a separable state. Then, from the linearity of the map, we have

$$\Phi_B(\rho_{AB}) = \alpha\Phi(\sigma_{AB}) + (1-\alpha)\Phi(\left|\psi\right\rangle\left\langle\psi\right|). \tag{5.1.2}$$

And, if our chosen entanglement measure E is convex

$$E(\Phi_B(\rho_{AB})) \leq \alpha E\left(\Phi_B(\sigma_{AB})\right) + (1-\alpha)E\left(\Phi_B(\left|\psi\right\rangle\left\langle\psi\right|\right) \tag{5.1.3}$$

$$= (1-\alpha)E(\Phi_B(\left|\psi\right\rangle\left\langle\psi\right|) \leq E(\Phi_B(\left|\psi\right\rangle\left\langle\psi\right|). \tag{5.1.4}$$

Here, we assumed that the measure of choice is zero or positive on separable states. We conclude that any mixed state of two qubits can always be outperformed by a pure state, for any channel present between them. In [81], this result was presented, but for the more general case where B has any dimensions. The optimal pure state, however, does not need to be a maximally entangled one for all channels, as was shown in [81, 82].

The question on whether there exists an indirect protocol that can improve the optimal direct protocol with pure states arises, and is still unanswered, to the best of my knowledge. Our attempts at a proof and sampling of protocols lead to believe that direct protocols with pure states are the most effective protocols, for any channel. However, although almost perfectly pure states can be prepared in a lab, they are not always cheap or available, due to the noise that may be present in the location where the state is prepared. In this case, mixed states might be cheaper overall.

5.2 Distribution through entanglement breaking channels is impossible

Our model of indirect distribution allows Bob to extract entanglement from a separable carrier that Alice sends (EDSS protocol). It is not completely unreasonable, then, to expect that, in some cases, this protocol for distribution could be effective even when the communication channel is entanglement breaking. However, we show that this is not the case, even indirect distribution is always ineffective if performed through any entanglement breaking channel. The reason is that the transformation that EBC performs can be described through a LOCC operation.

We prove the following theorem:

Theorem 3. *In any indirect protocols where the communication channel is entanglement breaking, the following inequality holds:*

$$E^{(f)}_{A:BC} \leq E^{(i)}_{B:AC}. \tag{5.2.1}$$

Proof. We first observe that the entanglement in the partition $A : BC$, after particle C goes through an EBC is always less than that of a classical state. The state after the channel can be written as in the right hand side of Eq. (2.1.7), with $X = A, B$:

$$\begin{aligned} E^{(f)}_{A:BC} =& E_{A:BC}\left(\sum_n Tr_C(|w_n\rangle\langle w_n|\,\rho_{ABC}) \otimes |v_n\rangle\langle v_n|_C\right) \\ \leq& E_{A:BC}\left(\sum_n Tr_C(|w_n\rangle\langle w_n|\,\rho_{ABC}) \otimes |V_n\rangle\langle V_n|_C\right), \end{aligned} \tag{5.2.2}$$

where $|v_n\rangle$ and $|w_n\rangle$ form two over-complete sets in the Hilbert space of C and $|V_n\rangle$ form a basis on a larger Hilbert space.

The reason why this holds is that you can always go from an over-complete, non-orthogonal to an orthogonal basis set through a local operation on C. Now note that since the $|V_n\rangle$ vectors are orthogonal, the state in the right hand side of Eq. (5.2.2) is classical, i.e. it has zero discord. By using the discord bound, given

in sec. 2.3.3,

$$|E_{A:BC} - E_{B:AC}| \leq D_{AB|C} = 0, \tag{5.2.3}$$

we see that the entanglement of such state is the same in the partitions $A : BC$ and $B : AC$:

$$\begin{aligned} &E_{B:AC}\left(\sum_n Tr_C(|w_n\rangle\langle w_n|\,\rho_{ABC}) \otimes |V_n\rangle\langle V_n|_C\right) = \\ &E_{A:BC}\left(\sum_n Tr_C(|w_n\rangle\langle w_n|\,\rho_{ABC}) \otimes |V_n\rangle\langle V_n|_C\right). \end{aligned} \tag{5.2.4}$$

Since you can always construct a POVM that goes from ρ_{ABC} to the state on the left hand side of Eq.(5.2.4), it also holds

$$E_{B:AC}\left(\sum_n Tr_C(|w_n\rangle\langle w_n|\,\rho_{ABC}) \otimes |V_n\rangle\langle V_n|_C\right) \leq E_{B:AC}(\rho_{ABC}) = E^{(i)}_{B:AC}. \tag{5.2.5}$$

By combining eqs.(5.2.2), (5.2.4), (5.2.5), we prove that the final entanglement between the laboratories is always bound by the initial one, as in the statement (5.2.1). Note that the initial entanglement $E^{(i)}_{B:AC}$ does not change if we allow a unitary operation on A and C, so this result extends to all the protocols that we described so far. □

5.3 Some distribution protocols in a noisy environment

The fact that EBC do not allow indirect distribution, however, does not mean that if another channel disrupts the entanglement of a particular carrier, the entanglement distribution will not be effective. Then, as one may expect, excessive protocols, where entanglement of the carrier can be small, may offer an advantage compared to non-excessive ones. We now explore this statement in some special cases of entanglement distribution in a noisy environment.

At first, we study some cases that show how the existence of excessive protocol

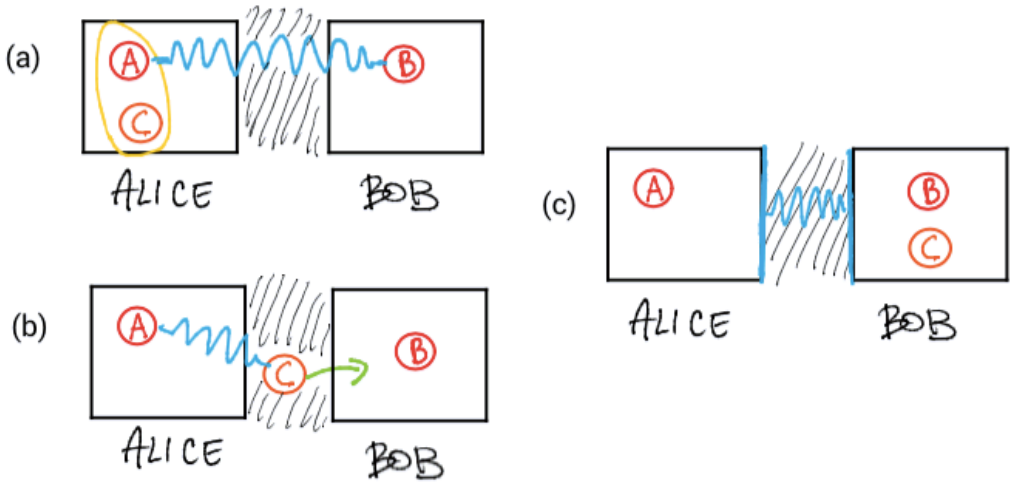

Figure 5.1: Indirect distribution of entanglement via noisy channel. (a) A and B are initially displaced and share some correlations when C interacts with A. (b) C is sent through a noisy channel to Bob. (c) The final entanglement that Alice and Bob share is $E_{A:BC}$.

allows the distribution of entanglement over a broad set of parameters of some particular states, whether the initial condition is of A and B being displaced or not. We then discuss how the presence of noise, not only in the channel, but also in the laboratories where the particles are created and stored could affect the distribution protocols.

5.4 Indirect protocol through a noisy channel

Our idea of indirect protocol of entanglement distribution through a noisy channel is represented in Fig. 5.1. Particles A and B are initially displaced, sharing an initial entanglement $E^{(0)}_{A:B}$ in the state ρ_{AB}. Particle C is completely uncorrelated to the rest of the system in its own state ρ_C, until Alice performs a unitary operation U_{AC} on particles A and C. At this point, we evaluate the initial entanglement as $E_{AC:B}$, in the state $\rho_{ABC} = U_{AC}(\rho_{AB} \otimes \rho_C)U^\dagger_{AC}$. Then, Alice sends C through the communication channel she shares with Bob, where the state of the system is affected by quantum noise acting on C. When C reaches Bob, the final entanglement

between the labs is $E_{A:BC}$ over the final state ρ^f_{ABC}, that is given by:

$$\rho^f_{ABC} = \sum_n K_n \rho_{ABC} K_n^\dagger. \tag{5.4.1}$$

Here the set of Kraus operators $\{K_n\}$ models the noise in the quantum channel and only acts on C. To represent these channels we use the three typical one-qubit channels, that we introduced in 2.1.2: the dephasing, the depolarizing and the amplitude damping channel.

Our protocol is similar to the one realized in the experiment described in chapter 3. In the protocol, we assume initially that C is uncorrelated to A and B and in the mixed state of the form

$$\alpha_C = \frac{1}{2}(\mathbb{I}_2 + s\sigma_x), , \tag{5.4.2}$$

where $s \in [-1, 1]$, while A and B are in a Werner state

$$\alpha_{AB} = p \left|\phi_+\right\rangle \left\langle\phi_+\right| + \frac{(1-p)}{4}\mathbb{I}_4. \tag{5.4.3}$$

In this state, A and B are entangled for $p > 1/3$. The two-qubit unitary operation that Alice has available in her lab is a controlled-phase gate, that acts on A and C as described in chapter 3. This operation happens to be, according to our numerical simulations, the entangling gate that allows the most entanglement to be distributed, in this particular protocol.

To show how the presence of noise in the channel affects the efficacy of this protocol, in Fig. 5.2 the negativity gain is plotted as a function of the noise of the state parameters. Also, the range of parameters in which the protocol is excessive is marked by the black line. In first row of plots (a) to (c), we allow the initial state of the carrier C to change its purity parameter, fixing the initial state of A and B. In this case, we observe the following properties of the protocol, regarding the advantages or disadvantages of excessive protocols:

- given a value of a noise parameter of any of the three channels, the protocol that allows maximal negativity gain is always the one where C starts in a pure

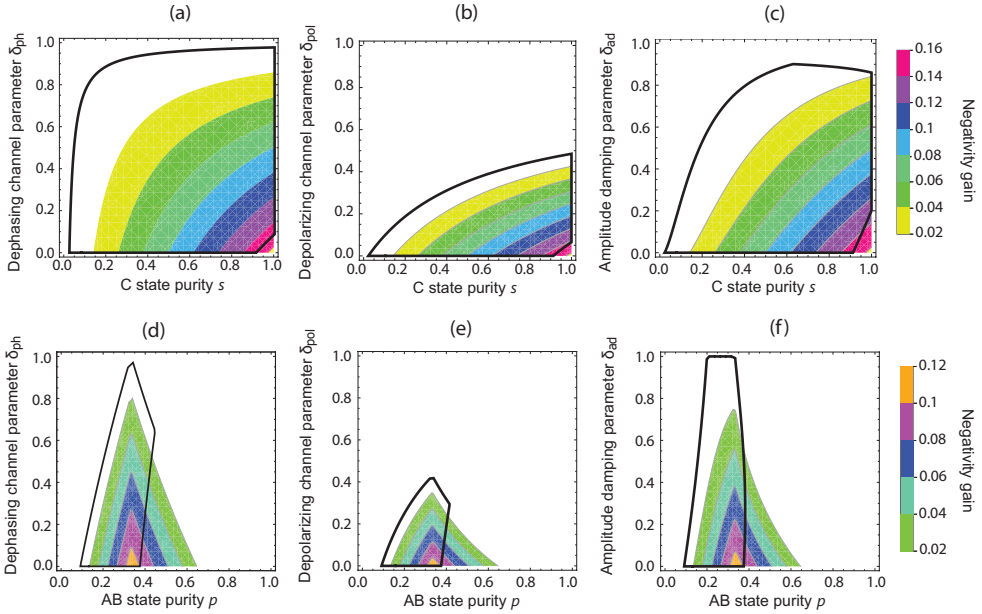

Figure 5.2: Negativity gain in the protocol depicted in Fig. 5.1. in the columns, the nature of the noise in the channel varies. Inside the thick, black line, the protocol is excessive. Particles A and B start in a Werner state, Eq. (5.4.3), while C in a mixed state of the form (5.4.2). The plots (a) to (c) present the gain as a function of the parameter of the carrier state, s in Eq.(5.4.2). In these plots, the entanglement parameter of the Werner state is fixed to $p = 0.34$, but qualitatively the same plots are obtained for different p. Notice that the optimal gain is reached if C is in a pure state. These cases lie outside the excessive region, but only for low noise. The lower row of plots, (d) to (f), present the negativity gain as a function of the purity of the A and B, p in Eq. (5.4.3). For these plots $s = 2/3$ but again, similar results are obtained for different values of s. Notice that the optimal protocol is for $p = 1/3$ and, for any noise, whenever it is effective, it is always excessive.

state, which is consistent with the result presented for direct distribution;

- the protocol is in the non-excessive region as long as the noise is weak (δ close to zero);

- if the noise is strong enough, the communicated entanglement is largely disrupted, and the optimal state of C falls in the excessive region. In this case, the existence of excessive protocols allows an effective distribution to occur even though a very noisy channel is used;

- the excessive protocol is verified for a very broad set of parameters, making

it easier to realize for Alice and hence more robust to the errors in the state preparation.

In the second row, (d) to (f), we complete the picture by fixing the state of C, and plotting the negativity gain when the purity of the Werner state parameter p is allowed to change. Here are some observations related to these plots:

- the region where excessive protocols occur is not as broad as in the previous instance;
- however, the protocol with the greatest increment is always, for any type of channel and noise parameter, well inside the excessive region;
- the protocols with the largest entanglement gain are the ones where the Werner state is at the "border of separability", for $p = 1/3$.

We conclude that, in this protocol, excessiveness offers an advantage if the noise is strong or if the initial state of A and B is fairly mixed and cannot be purified.

5.5 Direct-Indirect protocol through a noisy channel

The protocol described so far assumes that particles A and B are already far from each other, yet prepared in an entangled state, or at least that they are sharing some form of quantum correlation. Achieving this initial configuration can be difficult, however. We are expecting Alice and Bob to be able to prepare this configuration through the noisy channel that separates their labs and in our scenario this channel cannot be avoided.

Fig. 5.3 depicts a protocol that includes the preparation of the initial configuration. Alice prepares A and B, in her own laboratory, in the Werner state, Eq. (5.4.3). Next, she sends qubit B, through the noisy channel, to Bob. This is the direct part of the distribution. Alice and Bob likely share some entanglement, or some other form of correlation at this stage. Now the indirect part of the distribution begins.

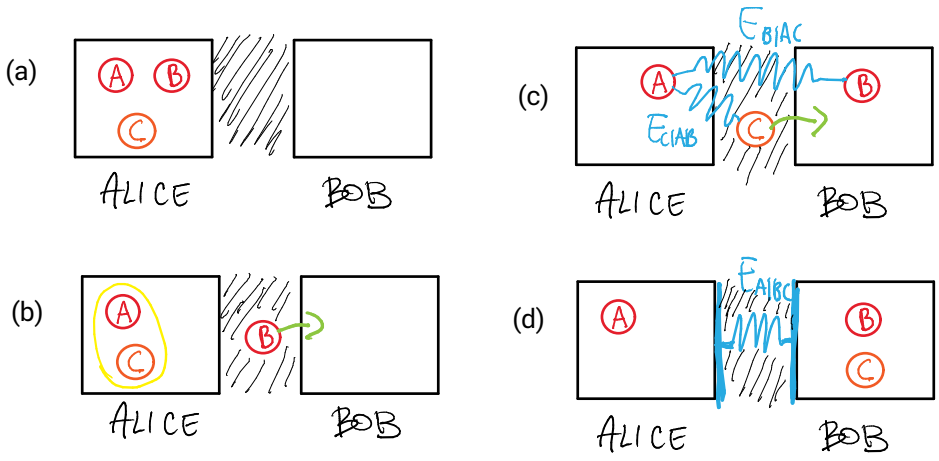

Figure 5.3: Direct distribution, followed by an indirect one, through a noisy channel. (a)Alice correlates A and B locally. (b) B reaches Bob through the noisy channel, and A and C interact with each other. (c) Finally C is sent to Bob, (d) increasing the entanglement shared with Alice.

Alice uses initial state of the qubit C, given in Eq. (5.4.2) and the c-phase operation, just as it was described in the previous section. This is the stage at which the initial entanglement is evaluated as $E_{B:AC}$

We now want to compare the effectiveness of the excessive protocols to the non-excessive ones, as we did before. But we would also like to see if, and in which cases, the indirect distribution is actually useful, when the direct one is present.

The considerations that arise by looking at plots (a) to (c) in fig. 5.4, where we vary the initial purity of C are very similar to the case with only indirect protocols shown in fig. 5.2. As before, we see that, whenever possible, having a pure C offers an advantage. The non-excessive region is, in this case, amplified. However, sufficiently high noise parameters in the channel make the optimal protocol excessive. We also observe that the excessive region is still significantly broader, for strong noises, than the non-excessive one. Note that these plots are realized by fixing $p = 0.34$. This value is very close to the border of separability of the Werner state. In nearly all the region of parameters shown, then, the state of B is separable after it reaches Bob.

The lower plots, (d) to (f) in fig. 5.4, show how having added the possibility of performing a second step in the distribution helps Alice and Bob achieving a highly

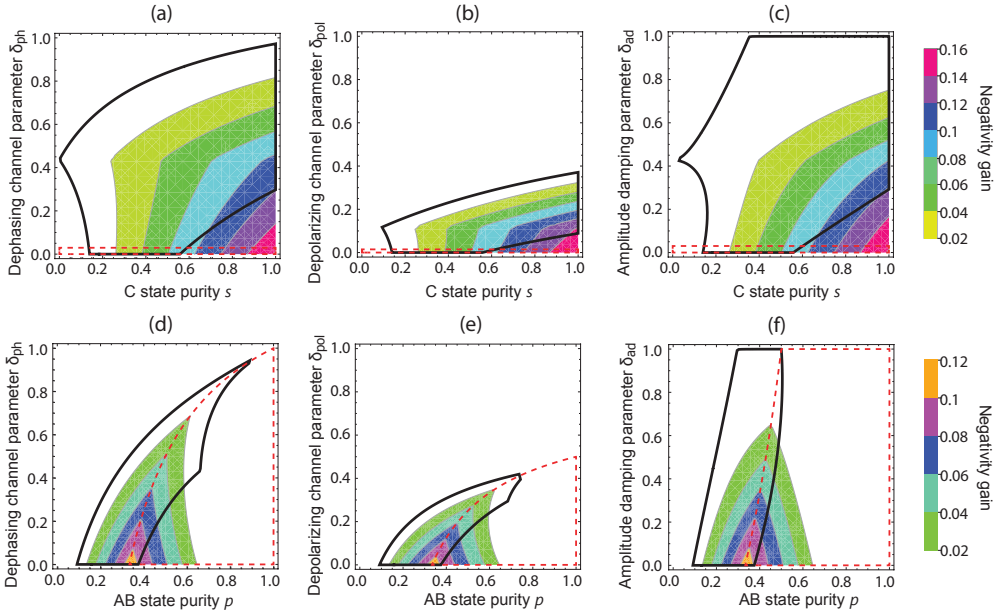

Figure 5.4: Gain in entanglement, measured as negativity, achieved in the indirect part of the distribution described in Fig. 5.3. The thick line contains the region where the protocol is excessive, the dashed line describes the region where a positive entanglement gain is achieved by only sending B (direct protocol). In the graphs (a) to (c), the parameter of the Werner state is fixed, $p = 0.34$, but the plots obtained for different p are qualitatively similar. Similarly to Fig. 5.2, the maximum gain is obtained with C prepared in a pure state. In the plots (d) to (f), the gain is presented as a function of the parameter of the Werner state of A and B, i.e., p in Eq. (5.4.3). C is initially in a mixed state, (5.4.2) with $s = 2/3$. Similar qualitative results are seen for different values of s. The largest gain is achieved with excessive protocols (whenever possible) and as in Fig. 5.2 the optimal value of p is at the border of separability of A and B (dashed line). If A and B are quite mixed ($p \leq 0.65$ for no noise), the indirect part of the protocol improves the direct distribution of entanglement, that is just impossible outside the dashed region.

entangled state. In these plots, we trace both the region of parameters where the indirect distribution is excessive (thick, black line) and the region in which the direct distribution is possible (dashed red line). The latter region is the one at which B successfully reaches Bob while still entangled.

As in fig. 5.2, the excessive distribution is the best protocol when the control on A and B is strong: the best value of p is always in the excessive region for any noise. Now, the exact value of p at which the distribution is optimal depends on the noise

parameter. However, it is still at the extreme of the region at which B is initially entangled (red dashed line).

Finally, observe that a significant entanglement gain is obtained with the indirect protocol, even in regions where the direct ones are not effective. Indirect distribution, then, can improve the number of cases where Alice and Bob can gain any entanglement at all, given that they have little control on C and on the channel that separates them. However, this advantage disappears if Alice is able to prepare A and B in a pure state, or in a nearly pure one (high p). This, together with other cases we studied, reinforces the idea that direct protocols using pure states are very hard, if at all possible, to beat. However, the presence of local noise in the lab could prevent Alice to effectively prepare a pure state, making indirect distribution a useful resource.

5.6 Entanglement distribution with local noise

As was illustrated in the previous examples, the purity of the initial states, of both the main parties A and B and of the carrier C, is an important factor in deciding which method of distribution is the most effective. It is interesting to study the case in which some local noise is present in Alice's and Bob's labs, apart from the one in the channel. Our protocol is still a direct one followed by an indirect distribution as shown in fig. 5.5, but now all the particles that are in Alice's lab are affected by a noisy channel. We again start from the Werner state of A and B, while C is initially in a pure state, $|x_0\rangle$. The local noise is modeled by an amplitude damping channel, while the noise in the channel, as in the previous sections, are described by dephasing, depolarizing, and amplitude damping channels.

By fixing $p = 0.34$, we obtain the plots in fig. 5.6. We see, as we would expect, that the presence of the local noise in both labs disrupts greatly the entanglement that is gained in the indirect process. However, the advantage of excessiveness in this case is striking: any gain in entanglement is obtained through an excessive distribution.

One may ask how many of these excessive protocols are with separable or entan-

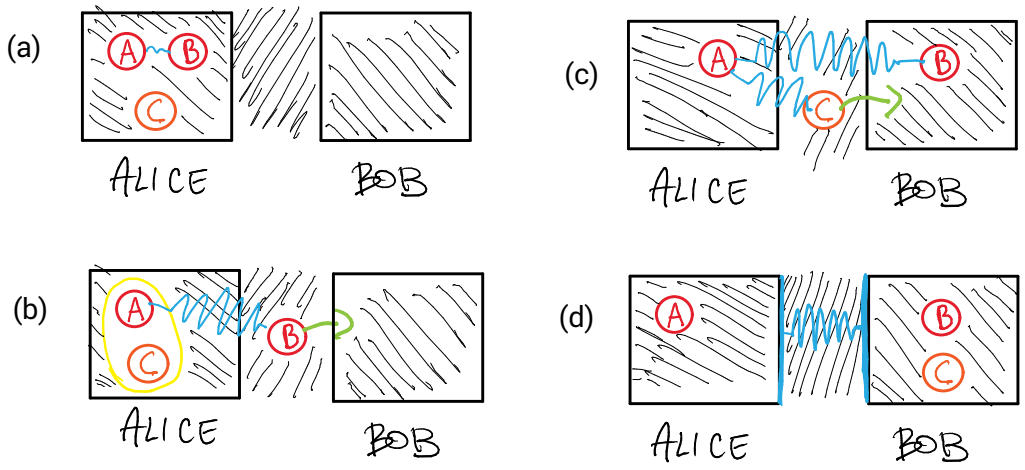

Figure 5.5: Direct distribution, followed by an indirect one, through a noisy channel, when local noise is also present in both labs. (a) Alice correlates A and B locally. (b) B reaches Bob through the noisy channel, and A and C interact with each other, and are also affected by one-qubit noise. (c) Finally C is sent to Bob, while A and B are again affected by the local noise (d) increasing the entanglement shared with Alice.

gled carriers. To answer this question, we numerically evaluated, for many values of p, the robustness of a class each protocols as the number of noise parameters, both in Alice's lab and in the channel, that allow the protocol to belong to that class. The robustness is defined as the areas of colored regions for similar plots to the one shown in fig. 5.7. The equivalent plots for dephasing and amplitude damping channels present very similar characteristics. However, a unique feature of the depolarising case is that the direct protocol can be improved, with a finally separable carrier, even if A and B are initially maximally entangled ($p = 1$).

Fig. 5.8 shows the area robustness of different protocols for different values of Werner state parameter p for depolarizing channel noise and local amplitude damping noise in Alice's lab. The equivalent plots for dephasing and amplitude damping channels present very similar characteristics. However, a different feature of the depolarising case is that the direct protocol can be improved, with a finally separable carrier, even if A and B are initially maximally entangled as is shown in fig. 5.8, where the dotted and dashed-dotted curves do not go to zero for $p = 1$.

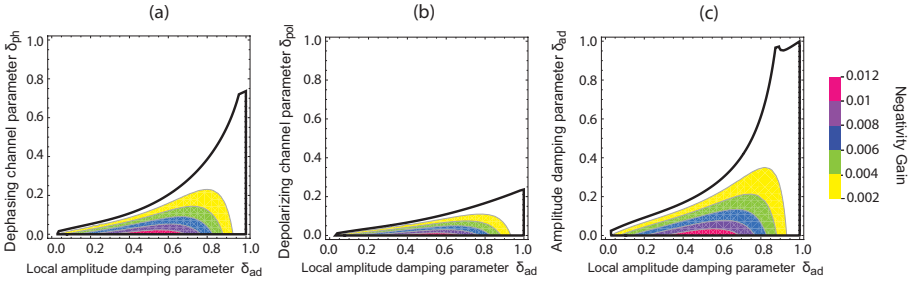

Figure 5.6: Negativity gain achieved in the indirect phase of the protocol described in Fig. 5.5. The parameter for the Werner state is $p = 0.34$, while C is prepared in a pure state. The local noises of Alice and Bob are modeled by amplitude damping channels and assumed to have the same parameter, plotted in the horizontal axis. On the vertical axes we show the strength of different noises in the channel. Within the thick line the protocol is excessive, meaning that the communicated entanglement is smaller than the one that is gained. The regions where the noise parameters generate an excessive protocol is very broad. The points with largest increment, for local noise parameter about $\delta_{\mathrm{ad}} \sim 0.5$, are always in the excessive region. As expected, the gain in negativity is greatly reduced compared to the case where the local noise is not present, in Fig. 5.4.

We see that the use of indirect protocols allows distribution for some sets of parameter for any value of p, and an increment in entanglement is observed in a very large region (blue, dashed-dotted curve). The direct distribution is impossible when the initial state is separable ($p < 1/3$), and in this case, indirect distribution is the only viable option. Even when direct distribution is effective ($p > 1/3$), the second, indirect, step makes the range of parameters in which distribution is successful much larger. The area for which a positive increment is obtained with a carrier that is separable in the end is smaller, but comparable, to the total area obtained from the excessive, indirect distribution.

5.7 Conclusions

We discussed how the presence of noise affects entanglement distribution protocols using qubits. If quantum noise is present in the communication channel, the best direct protocols involve pure states. When indirect protocols are included, however, the idea is that they could offer some advantages, because the partial, or even total, loss of communicated entanglement is not as "fatal" for this class of protocols.

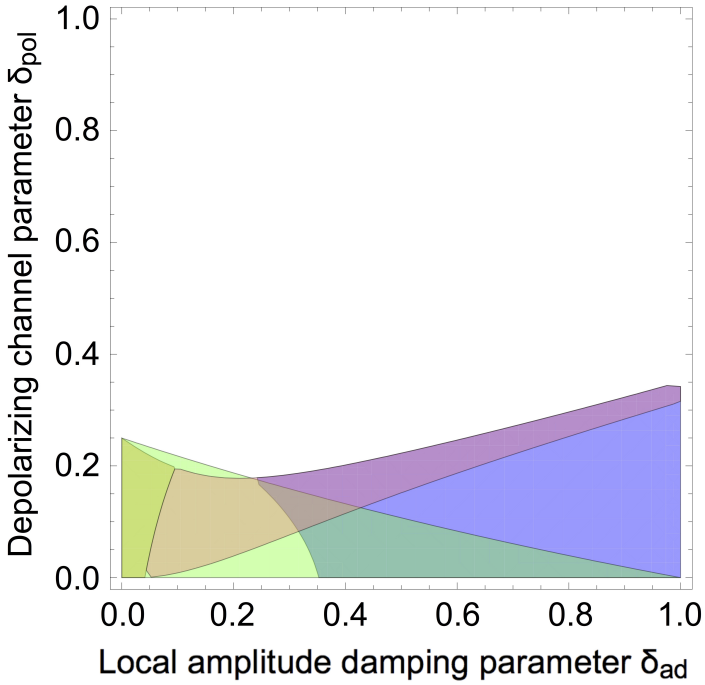

Figure 5.7: Visual representation of different entanglement distribution protocols, starting with a Werner state with parameter $p = 1/2$. In the vertical axes we show the depolarising channel parameter, in the horizontal one, the local amplitude damping noise parameter. In the green region, the simple direct protocol is effective. In the blue region, the indirect protocol produces a negativity increment. The pink and orange region represent the cases where the carrier is separable when it reaches Bob, and leaves Alice's lab, respectively, as entangled or as separable.

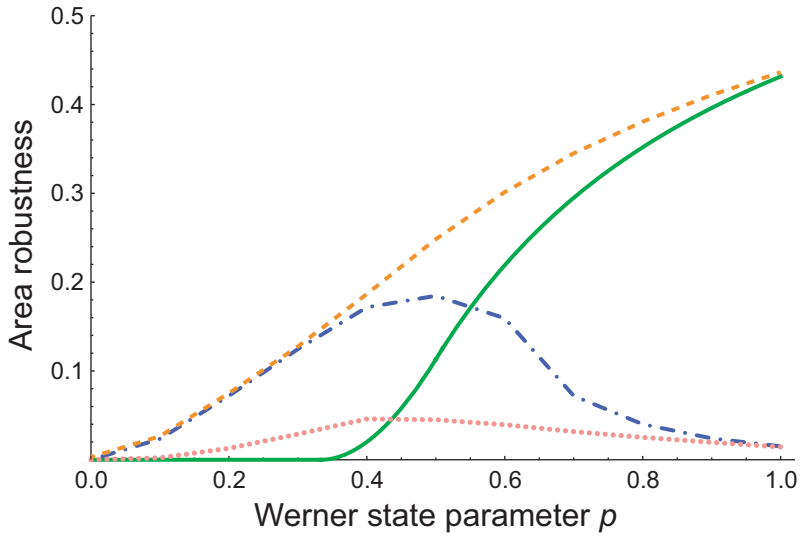

Figure 5.8: Area robustness of various protocols (vertical axis) as a function of the Werner state parameter p (horizontal axis) for local amplitude damping and a depolarizing channel noise. The curves are obtained by integrating over the regions, as in Fig. 5.7. Although the direct distribution is possible only for $p > 1/3$ (green full line), by using indirect protocols (orange dashed line), the creation of entanglement is possible for any value of p. Furthermore, the area on which we have an increment in entanglement after the indirect protocol (blue dashed-dotted line) is comparable for many values of p to the total area of entanglement distribution, and a large part of this area is with a finally separable carrier (pink dotted line).

However, in our study and protocol sampling, we didn't find an example of an indirect protocol that could outperform the optimal direct one, that is the one that uses pure states. Indirect protocols are not effective if the communication channel is entanglement breaking, offering no advantage in this scenario. On the other hand, if the purity of the initial state is compromised by external conditions, such as the presence of local noise or noise in the source, indirect protocols, particularly excessive ones, allow to increase the entanglement or even to generate entanglement in conditions where it would otherwise be impossible.

6. Increasing entanglement via measurements with unknown results

A global projective measurement can increase the entanglement of a bipartite state. For example, measuring a state of two qubits on a Bell basis and then reading the result (post-selecting) projects it onto a Bell state, that is always more entangled than the initial state, unless this is another Bell state. However, if we do not post-select, it becomes a different problem. We are trying to characterize the properties of the state and the basis that allow entanglement to grow in such an operation. This is not a distribution protocol, since it requires a global operation. However, it could in principle be used in the preparation of the entangled state used in direct distribution protocols, although it does not necessarily offer an advantage compared to a global unitary gate.

We want to study the conditions that allow entanglement to grow while applying a projective measurement on an initially pure state, without post-selecting the result. This is still work in progress, but some interesting results, both numerical and analytical are already available.

The states of the basis need to be entangled, at least as entangled as the initial state. However, the entanglement of the basis is not the only condition for increasing the final entanglement. We show this in two ways:

- the measurement on a Bell basis, without post-selection, cannot increase the entanglement of any state;
- if some of the states in the basis are separable, the final entanglement can be larger than the case where all the states are entangled.

6.1 Convex measures

We start from a pure state of a bipartite system AB, $|\phi_i\rangle$. The dimension of the Hilbert space is $D = d_A \times d_B$. This state has some entanglement, in a given measure E, given by $E(|\phi_i\rangle\langle\phi_i|) = E_i$. Next we measure an observable M given by $M = \sum_{k=1}^{D} \lambda_k |\psi_k\rangle\langle\psi_k|$. Each of the states in the basis $\{|\psi_k\rangle\}$ has entanglement $E(|\psi_k\rangle\langle\psi_k|) = E_k$, and for convenience, we assume $E_k \geq E_{k+1}$. If we do not post-select, the state after the measurement will be

$$\rho_f = \sum_{k=1}^{D} p_k |\psi_k\rangle\langle\psi_k| , \tag{6.1.1}$$

where $p_k = |\langle\phi_i|\psi_k\rangle|^2$. We ask if $\Delta = E_f - E_i = E(\rho_f) - E(|\phi_i\rangle\langle\phi_i|) > 0$. For any convex entanglement measure, and any dimensions of A and B, the entanglement of the most entangled state of the basis $|\psi_1\rangle$ bounds the final entanglement:

$$E(\rho_f) = E\Big(\sum_{k=1}^{D} p_k |\psi_k\rangle\langle\psi_k|\Big) \leq \sum_{k=1}^{D} p_k E\Big(|\psi_k\rangle\langle\psi_k|\Big) \leq \sum_{k=1}^{D} p_k E\Big(|\psi_1\rangle\langle\psi_1|\Big) = E\Big(|\psi_1\rangle\langle\psi_1|\Big). \tag{6.1.2}$$

We conclude that, if the measure of our interest is convex, a positive increment is only possible if at least one of the vectors of the basis is more entangled than the initial state. This result is very reasonable, entanglement is not created 'for free'.

6.2 Two-qubit states

6.2.1 Measurement on a Bell basis

We prove that, for any pure state of two qubits, a measurement on a Bell basis can never increase the entanglement, measured by the negativity. A general pure state can be decomposed on the Bell basis as

$$|\phi_i\rangle = A|\phi_+\rangle + B|\phi_-\rangle + C|\psi_+\rangle + D|\psi_-\rangle . \tag{6.2.1}$$

Without losing generality, we assume that $|A|^2 \geq |B|^2 \geq |C|^2 \geq |D|^2$. The state after the measurement is a Bell diagonal state

$$\rho = |A|^2 \, |\phi_+\rangle\langle\phi_+| + |B|^2 \, |\phi_-\rangle\langle\phi_-| + |C|^2 \, |\psi_+\rangle\langle\psi_+| + |D|^2 \, |\psi_-\rangle\langle\psi_-| \, . \tag{6.2.2}$$

As shown in sec. 1.3.2, the final negativity is $N_f = |A|^2 - \frac{1}{2}$. The negativity after partial transposition of the state $|\phi_i\rangle\langle\phi_i|$ can be evaluated as

$$N_i = \frac{1}{2} \left| A^2 - B^2 - C^2 + D^2 \right| \tag{6.2.3}$$

For any two complex numbers x and y, it holds $|x - y| \geq |x| - |y|$, as can be shown by applying the triangular inequality to $|x| = |x - y + y|$. Consequently $|x + y| = |x - (-y)| \geq |x| - |y|$. By using both these inequalities in eq. (6.2.3), we obtain

$$\begin{aligned} N_i = \frac{1}{2} \left| A^2 - B^2 - C^2 + D^2 \right| &\geq \frac{1}{2} \Big(|A^2 + D^2| - |B|^2 - |C|^2 \Big) \\ &\geq \frac{1}{2} \Big(|A|^2 - |D|^2 - |B|^2 - |C|^2 \Big) = |A|^2 - \frac{1}{2} = N_f, \end{aligned} \tag{6.2.4}$$

which shows that the negativity can only decrease in this process, i.e $N_f \leq N_i$.

6.2.2 Measurements on constant bases

The initial state of two qubits, in its Schmidt basis reads

$$|\phi_i\rangle = a \, |00\rangle + b \, |11\rangle \, , \tag{6.2.5}$$

with $a^2 + b^2 = 1$, and a and b being two real positive numbers between 0 and 1. The negativity of this state is given by $ab = a\sqrt{1 - a^2}$. We choose to take a measurement on a basis that has the property that all the states have the same Schmidt basis.

The states of such a basis can be uniquely written as

$$\begin{aligned}
|\psi_1\rangle &= c\,|00\rangle + d\,|11\rangle\,;\\
|\psi_2\rangle &= d\,|00\rangle - c\,|11\rangle\,;\\
|\psi_3\rangle &= f\,|01\rangle + \sqrt{1-f^2}\,|10\rangle\,;\\
|\psi_4\rangle &= \sqrt{1-f^2}\,|01\rangle - f\,|10\rangle\,,
\end{aligned} \tag{6.2.6}$$

where $d = \sqrt{1-c^2}$. The negativity of the first two states of the basis is $N_b = cd = c\sqrt{1-c^2}$. Without losing any generality, we can assume that $a^2 \leq 1/2$, $c^2 \leq 1/2$. We call this a 'constant basis', as the Schmidt basis is constant in all the basis.

We now the state in eq. (6.2.5) on this basis, obtaining the rank-2 final state $\rho_f = p_1\,|\psi_1\rangle\langle\psi_1| + p_2\,|\psi_2\rangle\langle\psi_2|$ that, whose partial transpose, in the Schmidt basis, reads

$$\rho_f^{T_A} = \begin{pmatrix}
d^2(bc-ad)^2 & 0 & 0 & 0\\
+c^2(ac+bd)^2 & & & \\
0 & 0 & cd((ac+bd)^2 & 0\\
 & & -(bc-ad)^2) & \\
0 & cd((ac+bd)^2 & 0 & 0\\
 & -(bc-ad)^2) & & \\
0 & 0 & 0 & c^2(bc-ad)^2\\
 & & & +d^2(ac+bd)^2
\end{pmatrix}, \tag{6.2.7}$$

The only negative eigenvalue of the partial transpose of this matrix gives rise to the negativity

$$N_f = \pm cd\left((ac+bd)^2 - (bc-ad)^2\right) = cd\left((a^2-b^2)(c^2-d^2) + 4abcd\right). \tag{6.2.8}$$

By inverting the equations $N_i^2 = a^2(1-a^2)$ and $N_b^2 = c^2(1-c^2)$, this expression can be rewritten as a function of the negativities of the initial state and of the basis as

$$N_f = N_b\left(\pm\sqrt{1-4N_i^2}\sqrt{1-4N_b^2} + 4N_iN_b\right). \tag{6.2.9}$$

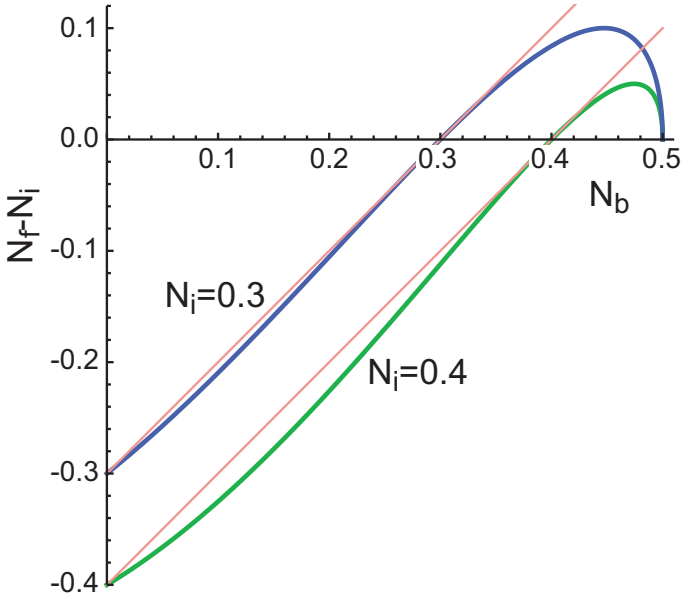

Figure 6.1: Negativity increment as function of the negativity of the basis, for two different values of N_i. The pink lines represent the upper bound, shown in the eq. (6.1.2) to be $N_b - N_i$.

The $\pm$ denotes the sign of the product $(c^2 - 1/2)(a^2 - 1/2)$. Per our assumption a^2 and c^2 are both smaller than $1/2$, therefore we have the plus sign. Notice that, in the other case, where the minus sign is obtained, the final negativity would turn out to be smaller. If $N_b = 0$ (i.e. the first two states of the basis are separable), we find that $N_f = 0$, for any initial negativity N_i. This has to be true, since in this case the final state in eq. (6.1.1) is already written in a separable form. If $N_b = 0.5$ (i.e. we measure over a Bell basis), on the other hand, we get $N_f = N_i$, as predicted by eq. (6.2.4). In both limits, then, the negativity does not grow in the process. We see, from plots similar to fig. 6.1, that these curves always have a maximum, unless the initial state is maximally entangled. By solving for N_b the equation $\frac{\partial N_f}{\partial N_b} = 0$, we find that the negativity of the basis corresponding to a maximum increment is

$$N_b^{\max} = \frac{\sqrt{2N_i + 1}}{2\sqrt{2}}. \tag{6.2.10}$$

By imposing the condition $N_f \geq N_i$, we find that we can obtain a positive increment

for measurement and states that satisfy

$$N_i < N_b < 0.5, \tag{6.2.11}$$

as is fig. 6.1 shows. Finally, by putting the value of the negativity of the optimal basis, eq. (6.2.10), into eq. (6.2.9), we find that the maximum increment that a measurement on such a basis allow decreases linearly with the initial negativity as

$$N_f^{\max} - N_i = \frac{1}{4} - \frac{N_i}{2}. \tag{6.2.12}$$

This shows that initially separable states can have the largest negativity gain ($N_f = 0.25$), when measured in a basis with negativity $\frac{1}{2\sqrt{2}}$, according to eq. (6.2.10). It also shows that any initial state has a strictly positive negativity gain in this process, unless it is maximally entangled.

6.2.3 Variation with local unitaries

So far in this section we assumed that the initial state had the same Schmidt basis as all the states in the basis. We can consider a more general case where we apply local unitaries U_A and U_B to the initial state, and measure on the basis in eq. (6.2.6). The final entanglement, in this case, is the same as if we applied the local unitary operation $U_A^\dagger \otimes U_B^\dagger$ to all the states of the basis.

We assume that $U_B = U_B^\dagger = \mathbb{I}$ and define the Schmidt basis of the initial state as

$$\begin{aligned} \left|\tilde{0}\right\rangle &= U_A \left|0\right\rangle = \sqrt{\lambda}\left|0\right\rangle + e^{i\phi}\sqrt{1-\lambda}\left|1\right\rangle; \\ \left|\tilde{1}\right\rangle &= U_A \left|1\right\rangle = \sqrt{1-\lambda}\left|0\right\rangle - e^{i\phi}\sqrt{\lambda}\left|1\right\rangle. \end{aligned} \tag{6.2.13}$$

In this way, the initial state is

$$\left|\phi_i\right\rangle = a\left|\tilde{0}0\right\rangle + b\left|\tilde{1}1\right\rangle, \tag{6.2.14}$$

while the basis is still the one in Eq. (6.2.6). We assume, without losing generality,

that $0 \leq \lambda \leq 0.5$.

When the Schmidt basis of the initial state and of the state of the basis were the same ($\lambda = 0$), the second two states of the basis $|\psi_3\rangle$ and $|\psi_4\rangle$ did not contribute to the final state. However, now they do contribute and the value of f affects the final entanglement.

First case: $f = c < \sqrt{0.5}$

To start, assume that $f = c$, i.e. all the states of the basis have the same entanglement. All the dependence on λ of the negative eigenvalue after the partial transposition is contained in an additive term $\lambda/2$ and a multiplicative term $(c^4(8\lambda - 4) + c^2(4 - 8\lambda) + \lambda^2)$. By comparing the expression of the final negativity with the one that is obtained for $\lambda = 0$, we can rewrite it as

$$N_f(\lambda) = -\frac{\lambda}{2} + N_f(\lambda = 0)\sqrt{1 - 2\lambda + \frac{\lambda^2}{4N_b^2}}. \tag{6.2.15}$$

$N_f(\lambda = 0)$ is given in eq. (6.2.9).

The derivative of eq. (6.2.15) with respect to λ is

$$\frac{\partial N_f}{\partial \lambda} = -\frac{N_f(\lambda = 0)\left(\frac{\lambda}{2N_b^2} + 1\right)}{2\sqrt{-\lambda - \frac{\lambda^2}{4N_b^2} + 1}} - \frac{1}{2} \leq 0. \tag{6.2.16}$$

This value is always negative, showing that the final negativity is a decreasing function of λ. These results are only valid for $0 \leq \lambda \leq 0.5$, but are symmetric for $0.5 \leq \lambda \leq 1$. We conclude that the final negativity, for a given value of N_b and N_i, is maximal when $\lambda = 0, 1$, i.e. when the Schmidt basis of the initial state and the measurement basis are parallel.

Second case: $f \neq c < \sqrt{0.5}$

As long as $c^2 \leq 0.5$, the condition $f < c$ implies that the entanglement of the first two states of the basis is larger than the one of the other two states. In this case, we take as the 'entanglement of the basis' the negativity of the two most entangled

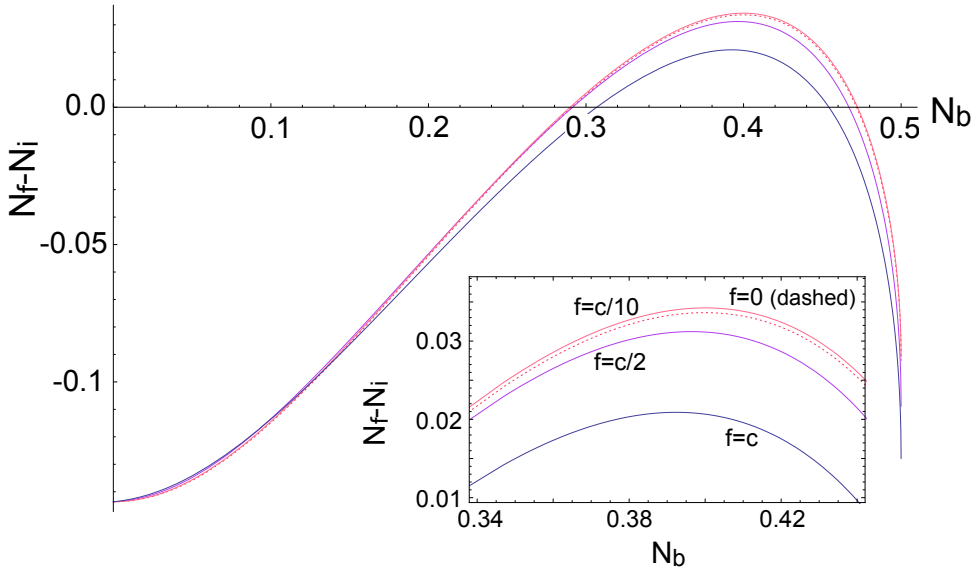

Figure 6.2: Negativity increment as a function of the negativity of the basis, for $N_i = 0.14$ and $\lambda = 0.2$. We see that the cases where $f < c$ give a larger negativity gain than $f = c$, although $f = 0$ is not optimal.

states, $N_b = c\sqrt{1-c^2}$. The case $f > c$ is equivalent, but with a complementary value of $\lambda' = 1 - \lambda$.

We plot the negativity increment as a function of N_b for many values of f. We see that, if the states of the basis are maximally entangled, the final entanglement decays. Quite surprisingly, we see also that the optimal value of f is not $f = c$ (maximum allowed negativity of all the states), but $f \approx c/10$. This means that a basis with little entanglement in some states is more 'effective' than one with more entanglement. On the other hand, the case where the two states with less entanglement in the basis are fully separable is still not the optimal one, although better than $f = c$. We also notice that the negativity increment starts to be positive way after the value of $N_b = N_i$.

6.3 General measurements

We consider a general basis, in the Schmidt basis of the most entangled state.

$$\begin{aligned}
|\psi_1\rangle &= c\,|00\rangle + d\,|11\rangle\,; \\
|\psi_2\rangle &= A_2\Big(d\,|00\rangle - c\,|11\rangle\Big) + B_2\,|01\rangle + C_2\,|10\rangle\,; \\
|\psi_3\rangle &= A_3\Big(d\,|00\rangle - c\,|11\rangle\Big) + B_3\,|01\rangle + C_3\,|10\rangle\,; \\
|\psi_4\rangle &= A_4\Big(d\,|00\rangle - c\,|11\rangle\Big) + B_4\,|01\rangle + C_4\,|10\rangle\,,
\end{aligned} \tag{6.3.1}$$

where the complex coefficients A_i, B_i and C_i must satisfy the normalization and orthogonality conditions $\langle\psi_i|\,\psi_j\rangle = \delta_{ij}$. We order the states of the basis decreasingly with their entanglement $E\Big(\,|\psi_1\rangle\langle\psi_1|\,\Big) \geq E\Big(\,|\psi_2\rangle\langle\psi_2|\,\Big) \geq E\Big(\,|\psi_3\rangle\langle\psi_3|\,\Big) \geq E\Big(\,|\psi_4\rangle\langle\psi_4|\,\Big)$, unless otherwise specified, and as we did before, we assume that $c^2 \leq 1/2$.

6.3.1 Measuring a separable state

We prove that, by measuring the initially separable state $|\phi_s\rangle = |11\rangle$, the final density matrix is always entangled. The state after the measurement is given by

$$\rho_f = \sum_{k=1}^{4} |\langle\psi_k|\,\phi_s\rangle|^2\,|\psi_k\rangle\langle\psi_k| = d^2\,|\psi_1\rangle\langle\psi_1| + c^2\sum_{i=2}^{4}|A_i|^2\,|\psi_i\rangle\langle\psi_i| = d^2\,|\psi_1\rangle\langle\psi_1| + c^2\rho_2, \tag{6.3.2}$$

where $\rho_2 = \sum_{i=2}^{4}|A_i|^2\,|\psi_i\rangle\langle\psi_i|$ is a density matrix, satisfying the normalization condition $\mathrm{Tr}(\rho_2) = 1$. We can apply any local unitary operations on this state, without altering the entanglement. We choose to apply the unitary operation $U_A = i\sigma_Y$ on qubit A. We obtain the matrix

$$\rho' = U_A\rho_f U_A^\dagger = d^2\,|\psi_1'\rangle\langle\psi_1'| + c^2\rho_2', \tag{6.3.3}$$

where $|\psi_1'\rangle = c\,|10\rangle - d\,|01\rangle$ and $\rho_2' = U_A\rho_2 U_A^\dagger$ is another density matrix.

In order to show that this state is entangled, we apply to it the swap operator

$$\Sigma = \sum_{i,j=0}^{1}|i\rangle\,\langle j|_A \otimes |j\rangle\,\langle i|_B. \tag{6.3.4}$$

This operator, as was discussed in sec. 1.2, is an entanglement witness, meaning that if

$$\mathrm{Tr}(\Sigma\sigma) < 0, \tag{6.3.5}$$

this means that the density matrix σ is entangled. We apply this criterion to the state in eq. (6.3.3). The action of the swap operator on a pure state is of inverting the states of A and B is

$$\Sigma\rho' = d^2\Big(c\,|01\rangle - d\,|10\rangle\Big)\Big(c\,\langle 10| - d\,\langle 01|\Big) + c^2 S\rho_2. \tag{6.3.6}$$

For any positive matrix M, the swap operator Σ satisfies $\mathrm{Tr}(MS) \leq \mathrm{Tr}(M)$, as can

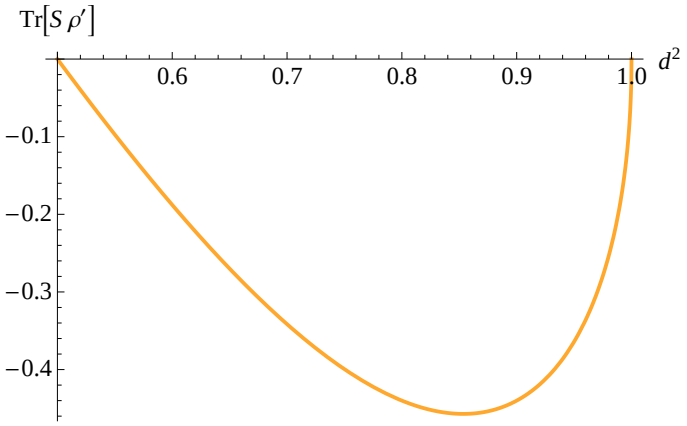

Figure 6.3: Upper bound to the average value of the swap operator Σ over the state ρ', $\mathcal{F} \equiv 1 - d^2\left(2\sqrt{1-d^2}d + 1\right)$, as a function of d^2. The plot shows that $\mathrm{Tr}(S\rho') < 0$ for any value of $0.5 < d^2 < 1$. This proves that the state is entangled.

be verified by taking the trace on the eigenbasis of M. Then

$$\mathrm{Tr}(\Sigma\rho') = -d^2(2cd) + c^2\mathrm{Tr}(S\rho_2) \leq -d^2(2cd) + c^2 < 0. \tag{6.3.7}$$

The second inequality holds, as shown in fig. 6.3, as long as $0.5 < d^2 < 1$. This proves that the state ρ', and consequently also the state ρ_f is entangled in this range. In the extreme values, the state $|\psi_1\rangle$ is either a Bell state ($d^2 = 0.5$) or a separable state ($d = 1$). Notice that we didn't use the assumption that the state $|\psi_1\rangle$ is the most entangled. Hence, this also proves that, for any measurement basis that has at least an entangled state that is not maximally entangled, there is at least a state that gains entanglement when measured over the basis.

6.4 Random basis sampling

We sampled 10 000 orthonormal bases of two qubits. We label as the entanglement of the basis N_b the negativity of the most entangled state, $|\psi_1\rangle$. The remaining states of the basis have in general different negativities and Schmidt bases, as in eq. (6.3.1). We perform a measurement on two initial states, with the same Schmidt

basis as $|\psi_1\rangle$, built as follows:

$$\begin{aligned} |\phi_s\rangle &= |11\rangle\,; \\ |\phi_i\rangle &= a\,|00\rangle + \sqrt{1-a^2}\,|11\rangle\,. \end{aligned} \tag{6.4.1}$$

where a is derived by inverting eq. (6.2.10) as

$$a = \sqrt{2\sqrt{-4c^8 + 8c^6 - 5c^4 + c^2} + \frac{1}{2}}, \tag{6.4.2}$$

In other words, the second state in eq. (6.4.1) is the one on which the basis would give raise to the maximum final negativity if the basis were of the form (6.2.6) (constant basis).

We now plot the negativity increment as a function of N_b, in figs. 6.4 and 6.5. We see in fig. 6.4 that all separable states $|\phi_s\rangle$ gain entanglement in this process, as we proved. Furthermore, their final entanglement can be larger for some bases that are not in the form (6.2.6), that reinforces the considerations made in the previous section: if the basis is non-optimal, less entangled states could be better for the negativity gain than more entangled ones.

In fig. 6.5, however, we see that the optimal constant basis with negativity given in eq. (6.2.10) allows the largest gain among all the bases with the same N_b that were sampled. This is reinforced by fig. 6.6, where it is seen that all the random bases on which the initial state was measured produce a smaller negativity gain than the bound in eq. (6.2.12) reached for a constant basis.

6.5 Conclusions

It is possible to increase the entanglement of a state by applying a projective measurement without post-selecting the result. Although the states of the basis need to be entangled for this process to be effective, the final entanglement does not grow with the entanglement of the basis. Particularly for two qubits, neither a separable nor a maximally entangled state can increase the negativity of a pure state.

For two qubits, we showed a derivation of the gain in negativity produced by

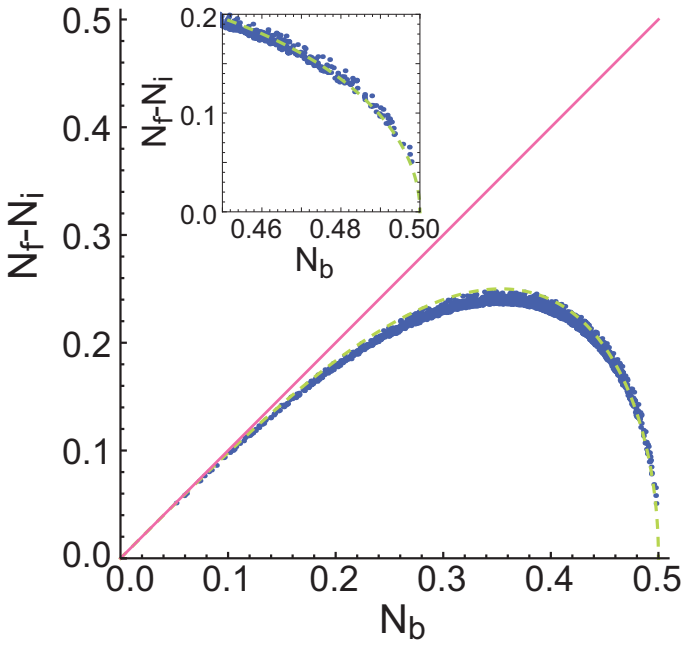

Figure 6.4: Negativity increment as a function of the negativity of the most entangled state of the basis that was randomly extracted (blue dots). The initial state is $|\phi_s\rangle = |11\rangle$, where both the local states belong to the Schmidt basis of the most entangled state in the measurement of the basis. In this case, some of the random non constant basis (blue dots) are better than the constant basis expressed in eq. 6.2.6 (green line), at least for high N_b. However, all the sampled bases succeed in increasing the entanglement after the measurement.

a 'constant basis', where all the states have the same Schmidt basis. Any non-maximally entangled state, with the same Schmidt basis, has an increment in negativity after the measurement. The maximum allowed gain is obtained for initially separable states, and decreases linearly with the initial negativity. We discussed how local unitary operations can affect the final negativity. We showed that a basis where two states have the same Schmidt basis as the initial state seems to be the best, among these sub-optimal measurements.

We proved that a separable state always gains entanglement with measurements on a basis that has one entangled state (that is not a Bell state) with a parallel Schmidt basis.

We also performed a random numerical sampling of the measurement basis. This, together with reinforcing the conclusions drawn in the previous sections, revealed,

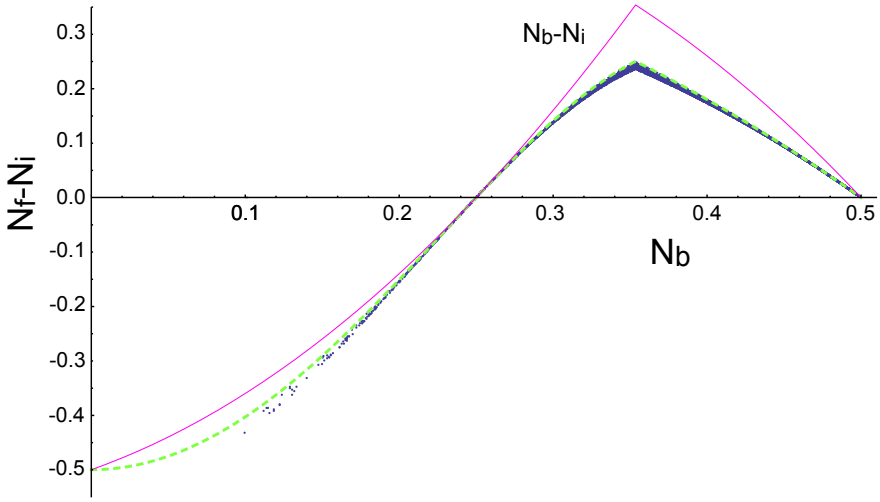

Figure 6.5: Negativity increment as a function of the negativity of the most entangled state of the randomly extracted basis. The initial state $|\phi_i\rangle$ is entangled and given in eq. (6.4.1). In this case, all the random non-constant-entanglement bases (blue dots) provide a lower gain than the constant basis expressed in eq. (6.2.6) (green line).

for a given state, the basis that allows the largest entanglement gain is a constant basis.

A full characterization of the properties of the measurement and of the initial state, allowing to produce entanglement gain is the goal of this work. However, this study is still work in progress. In the near future, it would be desirable to prove analytically the statements that the numerics suggested, and then to extend at least some of these considerations to bipartite states of higher dimensions.

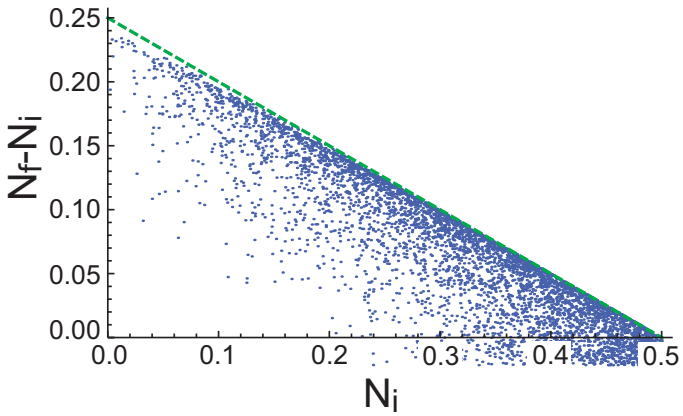

Figure 6.6: The entanglement increment given by a basis of the form given in equation (6.2.6), expressed as eq. (6.2.12) (green dashed line) is the largest among 10 000 measurement bases that were sampled (blue dots)

7. Outline and open problems

We studied some properties of entanglement distribution protocols, particularly in the distinction between direct and indirect distribution.

In chapter 3, we described the realization of a protocol for entanglement distribution that uses a separable carrier.

In chapter 4, we introduced a classification of distribution protocols, and showed some properties and examples of the various classes.

In chapter 5, we explored the advantages of some protocols over others, in the presence of noise, and depending on the external conditions. Direct protocols are the simplest class of entanglement distribution, and possibly they cannot be outperformed by indirect ones, if there is perfect control over the state, particularly if any pure state can be implemented. However, if pure states are not feasible, in some conditions indirect protocols, especially excessive ones, offer an advantage and even make the distribution possible.
Although almost pure states were created under laboratory conditions, they could be sometimes difficult to control, and part of the purity is easily lost in time. My impression is that indirect protocols, that are robust to mixedness in the state preparation and to the noise in the channel, could offer a cheap but less effective alternative to direct distribution with pure states.

In chapter 6, we described a class of global operations that can increase bipartite entanglement, using a projective measurement on a given basis without post-selection. We explored, both analytically and numerically, what are some properties of such basis that allow the entanglement of two qubits to grow in this process.

7.1 Open problems

The work presented in this thesis is largely confined to protocols where qubits interact for a finite fraction of time, ideally instantaneously, and is quite far from being general.

7.1.1 Distribution in the presence of noise

A "complete picture" of entanglement distribution protocols, complementing the results that were mentioned in sec. 2.3.3, is a desirable goal. Particularly, a definitive proof of the optimality of direct protocols is missing. A larger description of the robustness to different noisy channels and a generalization of the results to higher dimensions, finite as well as continuous, is another unachieved goal of this research.

7.1.2 Distribution with continuos interactions

Distribution protocols using continuos interactions are quite unexplored. A work on this scenario has recently been submitted [107]. These protocols present some characteristics that are quite different from the discreet protocols described in this thesis. A study on the limits and advantages that they offer, in the presence of noise or without it, and a comparison to the results that are already known for discreet interaction would be very interesting. A way of distributing entanglement that is somewhat unexplored is the one where A and B are interacting, for a continuos time interval, not with each other, but with a common mediator C. The interaction Hamiltonian describing the dynamics of such system would have the form

$$H = H_{AC} + H_{BC}. \tag{7.1.1}$$

In Cubitt *et al.*'s paper [73], it was already shown that these protocols can work even with a separable mediator. However, other questions arise, such as if discord is still a bound to distributed entanglement, or at least a necessary resource for such distribution scheme to work.

We are still investigating the answer to this question, and it's not very trivial. According to the Trotter expansion, the evolution of the state can be written

$$\rho_{ABC}(t) = \lim_{n\to\infty} (e^{-iH_{AC}t/n} e^{-iH_{BC}t/n})^n \rho_{ABC}(0) (e^{iH_{AC}t/n} e^{iH_{BC}t/n})^n \tag{7.1.2}$$

that reveals repeated discrete distribution protocols where C interacts in turn with A and B, making them entangled. Applying the discord bound (2.3.10) to the state, we find that for short times δt

$$E_{A:BC}(\delta t) - E_{A:BC}(0) \leq D'_{AB|C}(\delta t) + D_{AB|C}(0), \tag{7.1.3}$$

where $D'_{AB|C}(\delta t)$ is the discord of the "virtual" state $\rho'_{ABC}(\delta t) = e^{-iH_{AC}\delta t} \rho_{ABC}(0) e^{iH_{AC}\delta t}$.

As a purely quantitative restriction, the above bound is rather trivial, because the left hand side is an infinitesimal quantity, bound from above by a finite number. However, it shows that, even though the discord is zero, $D_{C|AB} = 0$, in a short time interval an increment in entanglement could still be possible, due to the "virtual" discord $D'_{AB|C}(\delta t)$. However, we showed that a constantly classically correlated mediator cannot increase the entanglement in the system. This property can be used to detect the classicality of the environment by only locally observing the system, similarly to what was proposed in [108, 109].

Other questions that are still open are on how the noise affects such processes and, in general, what are the properties of the Hamiltonian and the initial states that allow entanglement to grow.

7.2 Other projects

7.2.1 Multipartite entanglement without multipartite correlations

Another project in which I take part is a follow-up on the work [110]. In this paper, it is shown that there exist states which contain genuinely N-partite entanglement,

but still have a zero N-partite correlation tensor. Furthermore, even though the tensor is identically zero, these states can still violate some Bell inequalities. The systematic construction given in [110] only works if N is an odd number, leaving the question open for an even number of parties. We proved that this class of states (entangled without correlations) does not exist for $N = 2$ qubits, but does for $N = 4, 6$. However, our numerics shows that these states need to have rank 3 and above.

List of publications

Relevant to this thesis

[T1] A. Fedrizzi, M. Zuppardo, G. G. Gillett, M. A. Broome, M. de Almeida, M. Paternostro, A. G. White, and T. Paterek, "Experimental distribution of entanglement with separable carriers", Phys. Rev. Lett. **111**, 230504 (2013);

[T2] M. Zuppardo, T. Krisnanda, T. Paterek, S. Bandyopadhyay, A. Banerjee, P. Deb, S. Halder, K. Modi, and M. Paternostro, "Excessive distribution of quantum entanglement", Phys. Rev. A **93**, 012305 (2016);

[T3] Krisnanda, Tanjung, Margherita Zuppardo, Mauro Paternostro, and Tomasz Paterek. "Revealing non-classicality of unmeasured objects." arXiv:1607.01140 (2016);

[T4] M. Zuppardo, S. Bandyopadhyay, A. Banerjee, S. Halder, and T. Paterek, "Increasing entanglement via measurements with unknown results", *in progress*.

Other publications

[T5] M. Zuppardo, J. P. Santos, G. De Chiara, M. Paternostro, F. L. Semiao, and G. M. Palma, "Cavity-aided quantum parameter estimation in a bosonic double-well Josephson junction", Phys. Rev. A **91**, 033631 (2015).

Quantum measurements

We now review the formalism of measurements on a quantum system, to which we referred several times in this thesis:

Projective measurements

Suppose that we have a system in a pure state $|\psi\rangle$, and we want to measure an observable M. This observable is diagonal in the basis $|m\rangle$, and has eigenvalues, corresponding to all the possible measurement outcomes, m:

$$M = \sum_m m\,|m\rangle\langle m| = \sum_m m\Pi_m, \tag{.1}$$

where we have renamed the projectors on the state $|m\rangle$, Π_m. The vectors $|m\rangle$ form a basis on the Hilbert space, therefore the projectors must satisfy

$$\begin{aligned} \mathrm{Tr}(\Pi_m\Pi_n) &= \delta_{mn}\Pi_m \\ \sum_m \Pi_m &= \mathbb{I} \end{aligned} \tag{.2}$$

The probability of measuring the value m is given by

$$p(m) = |\,\langle\psi|m\rangle\,|^2 = \langle\psi|\,\Pi_m\,|\psi\rangle = \mathrm{Tr}(\Pi_m\,|\psi\rangle\langle\psi|). \tag{.3}$$

The state after the m eigenvalue is measured is

$$|\psi_m\rangle = |m\rangle = \frac{\Pi_m\,|\psi\rangle}{\sqrt{p(m)}}. \tag{.4}$$

The average outcome of the measurement is given by

$$\langle M \rangle = \sum_m m p(m) = \langle \psi | M | \psi \rangle = \mathrm{Tr}(M |\psi\rangle\langle\psi|). \tag{.5}$$

Now, if the initial state is mixed

$$\rho = \sum_j p_j |j\rangle\langle j| , \tag{.6}$$

all the above quantities have to be evaluated on the pure states $|j\rangle$, and then averaged according to the probabilities p_j. For instance, the probability of measuring the outcome m is

$$p(m) = \sum_j p_j \mathrm{Tr}(\Pi_m |j\rangle\langle j|) = \mathrm{Tr}(\sum_j p_j \Pi_m |j\rangle\langle j|) = \mathrm{Tr}(\rho \Pi_m), \tag{.7}$$

where we used the linearity of the trace.

The state after the measurement is still pure:

$$\rho_m = |m\rangle\langle m| = \Pi_m = \frac{\mathrm{Tr}(\rho \Pi_m)}{p(m)}. \tag{.8}$$

and the average measured value is

$$\langle M \rangle = \mathrm{Tr}(M\rho). \tag{.9}$$

If we do not *post-select*, meaning that we do not look at the result of the measurement, the state in the end is a mixture of the possible states ρ_m with weights given by

$$\rho = \sum_m p(m) \rho_m = \sum_m \mathrm{Tr}(\rho \Pi_m) |m\rangle\langle m| . \tag{.10}$$

POVM

Positive-operator valued measure (POVM) is a generalization of the projective measurements. We use the treatment of Peres [111]. Suppose that the system is bipartite, and in the state $\rho_{AB} = \rho_A \otimes \rho_B$. We measure an observable on the Hilbert space of A and B:

$$M_{AB} = \sum_m m \left|m\right\rangle\!\left\langle m\right| = \sum_m m\Pi_m, \tag{.11}$$

The probability to find the outcome m in a measurement is

$$p(m) = \mathrm{Tr}(\rho_{AB}\Pi_m) = \sum_{(i,k)_A,(j,l)_B} \left\langle ij\right| \Pi_m \left|kl\right\rangle \left\langle k\right| \rho_A \left|i\right\rangle \left\langle l\right| \rho_B \left|j\right\rangle = \sum_i \left\langle i\right| A_m \rho_A \left|i\right\rangle = \mathrm{Tr}(\rho_A A_m), \tag{.12}$$

where we defined the operators,

$$A_m = \sum_j \left\langle j\right| \Pi_m \rho_B \left|j\right\rangle . \tag{.13}$$

The state of A after the value m was observed is

$$\rho_m = \frac{\mathrm{Tr}(\rho_A A_m)}{p(m)}. \tag{.14}$$

The hermitian operators A_m, that act on the Hilbert space of A, are elements of a POVM. They are positive, as can be shown from $\mathrm{Tr}(\rho_{AB}\Pi_m) \geq 0$, for any m and ρ_{AB}. Also, they satisfy

$$\sum_m A_m = \mathbb{I}_A. \tag{.15}$$

Reduction criterion of separability and limits for a class of distillation protocols In this sense, they generalize the set of operators Π_m for a projective measurement. However, they do not necessarily commute, nor are they orthogonal with each other. They are an over-complete set, in the sense that, in general, their number is higher than the dimension of A.

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