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BELL'S INEQUALITY IN TIME

Lamia Varawala

Thesis Guide:
Asst. Prof. Tomasz Paterek
Nanyang Technological University, Singapore

External Examiner's Signature:_____

Thesis Guide's Signature:_____

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Abstract

Quantum mechanics successfully explains (and makes further predictions about) phenomena classical mechanics could not accommodate within its framework. Bell's inequality provides a means of experimentally disqualifying the local hidden variables theory that generalises the realm of classical mechanics. However, it is derived for measurements that are spacelike separated events on multiple particles.

The objective of this project is to obtain a similar distinction but when measurements are carried out on a single particle at different instances in time. The temporal scenario is shown to be fundamentally different from the spatial scenario.

Thereafter, specific problems are investigated:

- It is shown that certain correlations that give rise to extreme violation of Bells inequalities to some degree can be explained classically in time.
- A method is proposed to obtain temporal Bell-like inequalities via spatial Bells inequalities with auxiliary communication.
- A problem is presented which cannot be perfectly solved by classical operations, yet it can be perfectly solved by classical measurements or a quantum protocol.

This work provides insight into the nature of measurement processes in time especially in terms of similarities and differences with processes in space.

Contents

Chapter 1

Introduction

1.1 State

Definition. *A **state** is the ordered set of variables belonging to a space, V, that uniquely defines a system.*

Information Content

Definition. ***Information** is possessed by a state by virtue of it having one out multiple possible values.*

The amount of information contained can be quantified in terms of number of bits, $l = \lceil \log_2 |V| \rceil$, required to represent the state.

Operation

Definition. *An **operation**, $U : V \to V$, on a system is one that alters the state.*

1.2 Measurement

Definition. *A **measurement** is the act of obtaining information from a system by investigating its state.*

Definition. *A **measurement setting** is what parametrises the input of the measuring apparatus.*

The measurement setting will be represented by s and the outcome by r. A set of settings and outcomes will then be represented as $\mathbf{s}$ and $\mathbf{r}$ respectively.

Probability

Given s, every r will have an associated probability that we represent most generally as $p(r|s;\lambda;\mu)$ where λ is the value of the variable state and μ is the value of a generalised variable that is available to the measuring apparatus that may affect the outcome.

1.3 Classical Mechanics

Classical Mechanics was the prevalent theory of mechanics till the turn of the nineteenth century. Some features of classical mechanics are as follows:

- The state λ may only be altered under an operation.
- $$p \in \{0, 1\} \tag{1.1}$$

  In other words, the outcome of a measurement is in-principle deterministic, given, of course, that we have information about all the parameters that may affect the outcome.
- A state in classical mechanics may thus be represented by its value.

1.4 Quantum Mechanics

Quantum Mechanics emerged as a theory to make up for the shortcomings of Classical Mechanics in being unable to explain observations such as the discrete spectra of atoms. Some features of quantum mechanics are as follows:

- States in quantum mechanics, represented as $|\psi\rangle \in \mathbb{C}^n$, lie in Hilbert space where n, which may be infinite, will be identified later, such that $\langle\psi|\psi\rangle = 1$, where $\langle\psi| = |\psi\rangle^\dagger$.
- Since the state lies in a continuous space, the system possesses an infinite amount of information.

- Since the state is normalised to have a unit length, $U^\dagger = U^{-1}$ or U must be unitary.
- Measurements are represented by operators. At the time of measurement, the state collapses instantaneously to an eigenvector of the corresponding operator. The outcome of such a measurement is the eigenvalue corresponding to the eigenvector.
- Since $|\psi\rangle \in \mathbb{C}^n$, there are only n distinct eigenvectors it can collapse to resulting in at most n distinct corresponding outcomes inspite of possessing an infinite amount of information.

$$l = \lceil \log_2 n \rceil. \tag{1.2}$$

- Since the state is changed at the time of measurement, measurements are invasive.
- $$p(r|s) = \sum_i |\langle \phi_i | \psi \rangle|^2 \in [0, 1], \tag{1.3}$$

  where $H|\phi_i\rangle = r|\phi_i\rangle$. In other words, the outcome of a measurement is indeterministic.

1.5 Qubit

Classical states can be represented using classical bits as variables. If quantum bits were attempted to be represented by classical bits, infinitely many bits would be required to do so. Therefore, it is useful to define a unit of quantum information instead, analogous to a classical bit.

Definition. *A **qubit** is a unit of quantum information* $|\psi\rangle \in \mathbb{C}^2$.

As expected, for a qubit, $l = 1$.

Physical Realisation

Qubits do not only serve as mathematical representations. They can be realised physically and incorporated into applications. For the purposes of this discussion, features and properties of qubits will be described keeping

electron spin in mind due to historical reasons. However, there are other realisations such as photon polarisation state that have features and properties and display effects isomorphic to electron spin (with regard to qualifying as a qubit) which may even be more feasible to work with experimentally.

Experimental observations in accordance with Quantum Mechanics predict that spin $1/2$ is an intrinsic property of particles such as electrons. Therefore, irrespective of what direction is chosen to measure electron spin, it will take one of two ($\pm 1/2 \hbar$) values qualifying it as a qubit.

Pauli Matrices

Pauli matrices are as follows:

$$\sigma_x = \begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}, \tag{1.4}$$

$$\sigma_y = \begin{pmatrix} 0 & -\iota \\ \iota & 0 \end{pmatrix}, \tag{1.5}$$

$$\sigma_z = \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \tag{1.6}$$

The properties of these matrices are as follows:

- They may represent measurements.
- They are unitary. Thus, they may represent operations.
- They obey the following commutation relations

  $$[\sigma_i, \sigma_j] = \iota \epsilon_{ijk} \sigma_k, \tag{1.7}$$

  where $1 \equiv x, 2 \equiv y, 3 \equiv z$. Thus, they qualify to represent angular momentum in quantum mechanics.
- Their spectrum is $\{-1, +1\}$ and they each have 2 orthogonal eigenvectors. Thus, they may correspond to a measurement with 2 possible outcomes.

These properties qualify Pauli matrices to represent spin measurements and operations on spin.

Spin in any arbitrary direction $\hat{s}$ can be represented as

$$\sigma_s = \sum_i \left(\hat{s} \cdot \hat{i}\right) \sigma_i. \tag{1.8}$$

By convention, we carry out measurements along z. The eigenbasis of σ_z is represented as follows:

$$|0\rangle = \begin{pmatrix} 0 \\ 1 \end{pmatrix}, \tag{1.9}$$

$$|1\rangle = \begin{pmatrix} 1 \\ 0 \end{pmatrix}, \tag{1.10}$$

where

$$\begin{aligned} \sigma_z|0\rangle &= +1|0\rangle, & (1.11)\\ \sigma_z|1\rangle &= -1|1\rangle. & (1.12) \end{aligned}$$

The eigenbasis will be used as qubits analogous to the values $\{0, 1\}$ of classical bits.

Pauli matrices may also be used as operators on the qubits with the following results:

$$\begin{aligned} \sigma_x|0\rangle &= |1\rangle, & (1.13)\\ \sigma_x|1\rangle &= |0\rangle, & (1.14)\\ \sigma_y|0\rangle &= \iota|1\rangle, & (1.15)\\ \sigma_y|1\rangle &= -\iota|0\rangle, & (1.16)\\ \sigma_z|0\rangle &= |0\rangle, & (1.17)\\ \sigma_z|1\rangle &= -|1\rangle. & (1.18) \end{aligned}$$

1.6 Entanglement

When pairs of electrons are created in a singlet state and measurements on the electrons are made individually along the same direction, the results obtained are perfectly anti-correlated in accordance with the requirement that angular momentum is conserved.

Definition. ***Entanglement*** *refers to the state of systems when the state of one cannot be described independently of that of the others.*

In quantum mechanics, such entangled electrons in a singlet state may be represented as

$$|\psi\rangle_{\mathrm{AB}} = \frac{1}{\sqrt{2}}(|0\rangle_{\mathrm{A}} \otimes |1\rangle_{\mathrm{B}} + |1\rangle_{\mathrm{A}} \otimes |0\rangle_{\mathrm{B}}), \tag{1.19}$$

where A and B are the measuring parties.

Paradox

A paradox[2] arises when we consider the invasive measurements of in quantum mechanics. Consider a measurement made at A. Upon making this measurement, $|\psi\rangle_{\mathrm{A}}$ collapses instantaneously to an eigenvector of the measurement operator. However, due to the form of the entangled state, this "forces" the measurement outcome at B to be the opposite.

If the measurements made at A and B are spacelike separated events, it is impossible, in the framework of special relativity, for A to communicate the measurement outcome to B. Inspite of that, the anti-correlation holds.

1.7 Local Hidden Variables Theory

Measurement outcomes in quantum mechanics are indeterministic. This feature resulted in a reluctance to accept the theory at a philosophical level. In addition, the paradox was sought to be resolved. This led to the development of the *Local Hidden Variables Theory*, which attempted to explain observations within a deterministic framework.

Definition. *A* ***hidden variable*** *is one that carries an arbitrary amount of information possessed by a system. The value of these variables cannot be controlled and come with an associated probability.*

The Local Hidden Variables Theory assumes that at the time of creation, systems possess such variables λ. The values of these variables can determine the outcome of a measurement.

The pair of electrons created at the same point in spacetime may share identical values of these variables thus eliminating the need for communication in order to obtain perfectly anti-correlated results.

1.8 Local Hidden Variables and Shared Randomness

Consider measurements made on spins of electrons in a singlet state by A and B in the framework of the Local Hidden Variables Theory. We have probabilities $p_{\mathrm{A}}(r_{\mathrm{A}}|s_{\mathrm{A}};\lambda)$ and $p_{\mathrm{B}}(r_{\mathrm{B}}|s_{\mathrm{B}};\lambda)$ associated with outcomes r_{A} and r_{B} respectively.

The only way we can investigate a process is by making measurements. Hence, it is only the measurement outcomes obtained given measurement settings that are of concern which are represented by p_{A} and p_{B}.

Definition. ***Shared randomness*** *refers to variables shared between measuring parties whose values, which have associated probabilities, can affect measurement outcomes.*

The same probabilities can be obtained by considering $p_{\mathrm{A}}(r_{\mathrm{A}}|s_{\mathrm{A}};\mu)$ and $p_{\mathrm{B}}(r_{\mathrm{B}}|s_{\mathrm{B}};\mu)$ instead where μ is shared randomness.

1.9 Bell's Inequality

Both, quantum mechanics and the local hidden variables theory seemingly have the same predictions and are distinct only at the philosophical level where the former adheres to logical positivism and the latter to logical realism.

However, if the restriction on measurement setting being the same for both parties while measuring electron spin is dropped and furthermore, parties are allowed to measure along any arbitrary direction, an inequality can be obtained which must hold if the local hidden variables theory is in play.

Correlation

We define correlation as follows:

$$E(\mathbf{s}) = \sum_{\lambda,\mu} p(\lambda)p(\mu) \sum_{\mathbf{r}} \prod_{i} r_i p_i(r_i|s_i;\lambda;\mu) \tag{1.20}$$

$$= \left\langle \prod_{i} r_i \right\rangle, \tag{1.21}$$

where any quantity enclosed within $\langle\cdot\rangle$ is averaged over.

This is general in that it accommodates multiple observers and accounts for both λ and μ but either one of the two may be considered.

For continuous variables, summation may be replaced with integration.

For electron spins where $r_{\mathrm{A}}, r_{\mathrm{B}} \in \{0,1\}$ and when $s_{\mathrm{A}}, s_{\mathrm{B}} \in [0,\pi) \otimes [0,2\pi)$, the local hidden variables theory must satisfy[1]

$$E(a,c) - E(b,c) - E(a,b) \leq 1, \tag{1.22}$$

which is obtained entirely on the basis of probabilistic considerations. It must be noted that due to considerations of locality, each party is unaware of the setting of the other. However, from (1.3), (1.8) and (1.19) it can be seen that quantum mechanics need not obey the inequality.

This is the celebrated *Bell's Inequality* which can be tested experimentally to show that it is infact violated showing that the local hidden variables theory fails.

Chapter 2

Hidden Variables in Time

Bell's inequality is derived for systems of entangled particles on which measurements are carried out at spacelike separations.

- It is used to draw a distinction between the classical realm of physics and quantum mechanics.
- It has guided experiments that demonstrate the need for quantum mechanics.
- It has formed the basis upon which superiority of quantum protocols over classical protocols is sought to be demonstrated.

The objective of this project is to obtain a similar distinction for processes taking place over time. The form of an inequality is desirable since they may accommodate experimental errors unlike strict equalities.

2.1 Contrast with Hidden Variables in "Space"

These processes that take place over time cannot be considered as a simple analogies to those taking place on entangled particles with spacelike separation in the following ways:

- For a process that takes place in time, a clear distinction between past and future must be made. This would require the separation between measurements to be timelike.

- The possibility of one-way signalling from past to future cannot be ignored since the separation between measurements is timelike.
- The measurements can be carried out on the same system. Hence, it is not required to use a source that produces entangled particles.
- Since measurements are carried out on the same particle, the most general protocol would allow classical measurements to also be invasive or equivalently, to couple measurements with operations.
- Due to invasiveness, the protocols must be considered as Markov chains when there are more than two timesteps unlike for spatially separated measurements which can be conducted at once independently.
- For spacelike separated measurements, locally the probability distributions are uniform (over a certain physically determined measure) in the case of both, quantum mechanics and the local hidden variables theory. However, in this case of temporal processes this assumption cannot be made due to invasiveness at every stage that may bias outcomes at the subsequent stages such that the locally obtained distribution may not be uniform. The distribution depends upon measurements and operations carried out in former timesteps. When comparing the outcomes obtained using different models, local probability distributions must also be matched.
- The inequality must be obtained for a more general hidden variables model than a local hidden variables model, more specifically one that accommodates one-way signalling (from past to future).

In the context of temporal correlations, let probability be expressed as

$$p_t(r_t|s_t) = p_t(r_t|s_t;\lambda_t;\mu) \tag{2.1}$$

and correlations as

$$E(\mathbf{s}) = \sum_{\lambda_1,\mu} p(\lambda_1)p(\mu) \sum_{\mathbf{r}} \prod_t r_t p_t(r_t|s_t;\lambda_t;\mu). \tag{2.2}$$

2.2 Literature Review

Several measures have been presented in literature that distinguish between classical and quantum mechanics in a temporal scenario. Some of them are as follows:

2.2.1 Leggett-Garg Inequality

The Leggett-Garg inequality[3] is a mathematical requirement placed on time-evolving systems that must be obeyed by classical theories.

It is shown to be violated by quantum mechanical processes.

Drawbacks

The inequality is derived based on the following that are assumed to hold for classical theories:

- Measurements are non-invasive. Any measurement made on a system does not affect its state.
- Macrorealism holds. The system under consideration has a definite state.

If the inequality does not hold, it may only disqualify theories for which these assumptions hold, not all classical theories.

2.2.2 Nonclassicality in Sequential Measurements

A protocol[4] is proposed in which general measurements are carried out sequentially on quantum systems.

It is shown that beyond a certain length of the sequence, the correlations cannot be obtained classically.

Drawbacks

However, there are certain drawbacks to this as follows:

- The set of parameters for which classical mechanics cannot reproduce quantum correlations is obtained analytically. However, if this has to be verified, millions of measurements will be required to be made. This is not feasible, even numerically.
- It is only the correlations on measurements that are reproduced. Local probability distributions on measurements are not necessarily reproduced if correlations are.

2.3 Infinite Information

When Bell's inequality was derived, no restriction was placed on the amount of information carried by λ. Therefore, it holds even when an infinite amount of information is carried.

Consider an infinite amount information carried as $\{\lambda_t \in [-1, 1]\}$, a set of hidden variables and $\hat{\lambda} \in [0, \pi) \otimes [0, 2\pi)$ which defines the state exactly in our problem of a sequence of measurements taking place on electron spin over time. The initial value of $\hat{\lambda}$ may be predefined.

We may allow the following protocol to act at every step:

$$r_t = \text{sgn}(\hat{\lambda} \cdot \hat{s}_t - \lambda_t) \tag{2.3}$$

and after the measurement has been made

$$\hat{\lambda} \to \hat{s}_t, \tag{2.4}$$

where $\hat{s} \equiv s \in [0, \pi) \otimes [0, 2\pi)$. From (2.1) and (1.8) it can be seen that this yields exactly the probability distributions that a qubit would yield and from (2.2) the correlations.

This is something we would expect intuitively since if λ is allowed to possess an infinite amount of information, it can simply exactly describe a quantum state which itself possesses infinite information.

Note that λ may still be considered as *hidden* as its initial value may follow a probability distribution on its range.

2.4 Difference between Hidden Variables and Shared Randomness

The reason a classical protocol with infinite information content in hidden variables cannot be disqualified in the case of a temporal sequence of measurements unlike that of spatially separated entangled systems is that the hidden variables carry information about the system which in this case has been altered by the measurements made and operations carried out in former timesteps. This has been exploited in the protocol suggested above where the system collapses according to the measurement made and thus carries information about the former setting which can be used to reproduce the quantum correlations.

In the case of temporal sequences of measurements on systems with hidden variables, the probability corresponding to a measurement outcome at time instance t is given by $p_t(r_t|s_t; \lambda_t(\lambda_{t-1}, s_t))$ which forms a Markov chain. Here, λ_t is not a constant even for a given instance of the protocol and thus cannot be replaced with μ which is. μ is a weaker source of information.

Therefore, when there is the possibility of signalling, a distinction must be made between shared randomness and hidden variables.

Definition. *At every stage, a function determines what the outcome should be given the measurement setting.* ***Shared randomness*** *is what expresses itself in determining what this function is out of all possible functions.*

This definition of shared randomness is equivalent to the one proposed before but provides a better defined physical manifestation.

Hidden variables will have the same definition as before.

2.5 Communication Cost of Simulating Quantum Correlations

If, in the case of spatially separated measurements on spin of electrons in a singlet state, all the shared information is attributed to shared randomness eliminating the need of a source, it proves to be useful to investigate what

happens when 1 bit of information is allowed to be communicated from A to B[5].

The shared randomness may contain an infinite amount of information and may thus be mapped to uniform probability distributions on $(\hat{m}_1, \hat{m}_2)$ which are unit vectors in 3D space. Let $\hat{a} \equiv s_{\rm A} \in [0, \pi) \otimes [0, 2\pi)$ and $\hat{b} \equiv s_{\rm B} \in [0, \pi) \otimes [0, 2\pi)$.

We may choose the following protocol.

1. A outputs $r_{\rm A} = -\operatorname{sgn}(\hat{a} \cdot \hat{m}_1) \in \{-1, 1\}$.
2. A communicates $\lambda = \operatorname{sgn}(\hat{a} \cdot \hat{m}_1) \operatorname{sgn}(\hat{a} \cdot \hat{m}_2) \in \{-1, 1\}$ to B.
3. B outputs $r_{\rm B} = \operatorname{sgn}\left[\hat{b} \cdot (\hat{m}_1 + \lambda \hat{m}_2)\right] \in \{-1, 1\}$.

Due to the uniform probability distribution on values of $(\hat{m}_1, \hat{m}_2)$, $\langle r_{\rm A} \rangle = \langle r_{\rm B} \rangle = 0$ as required by (1.3) and (1.8).

$$\langle r_{\rm A} r_{\rm B} \rangle = \left\langle -\operatorname{sgn}(\hat{a} \cdot \hat{m}_1) \operatorname{sgn}\left[\hat{b} \cdot (\hat{m}_1 + \lambda \hat{m}_2)\right] \right\rangle \tag{2.5}$$

$$= \left\langle -\operatorname{sgn}(\hat{a} \cdot \hat{m}_1) \times \sum_{d=\pm 1} \frac{1 + d\lambda}{2} \operatorname{sgn}\left[\hat{b} \cdot (\hat{m}_1 + d\hat{m}_2)\right] \right\rangle \tag{2.6}$$

Using the symmetries $\hat{m}_1 \leftrightarrow \hat{m}_2$ and $\hat{m}_2 \leftrightarrow -\hat{m}_2$

$$\langle r_{\rm A} r_{\rm B} \rangle = \left\langle 2 \operatorname{sgn}(\hat{a} \cdot \hat{m}_1) \operatorname{sgn}\left[\hat{b} \cdot (\hat{m}_2 - \hat{m}_1)\right] \right\rangle \tag{2.7}$$

$$= -\hat{a} \cdot \hat{b} \tag{2.8}$$

which is the expected quantum correlation from (1.3), (1.8), (1.19) and (1.20). This shows that the quantum correlations can be obtained exactly if there is shared randomness at the communication cost of 1 bit.

This investigation loses the constraint of locality by allowing for one way communication making it directly applicable to our problem where the communicated bit that carries information about the setting is equivalent to λ that is propagated in time.

In the case of measurements of spin on an electron propagating in time, from (1.3), (1.8) and (1.20) we require

$$\langle r_{\rm A} r_{\rm B} \rangle = \hat{a} \cdot \hat{b} \tag{2.9}$$

This can be achieved by

$$r_{\rm A} = {\rm sgn}(\hat{a} \cdot \hat{m}_1) \tag{2.10}$$

leaving the rest of the protocol as it is. The local probability distributions are retained.

The reason for making this investigation for the case of spatially separated measurements first is to highlight the important distinction between shared randomness and hidden variables. Inspite of allowing an infinite amount of shared randomness, the quantum correlations in space can be obtained only upon allowing 1 extra bit of information to be communicated. This would not be evident if the quantum correlations in time were to be considered directly for which no inequality like Bell's inequality has been introduced.

2.6 Information Content

We wish to obtain a measure that would tell a phenomenon that cannot be realised within the classical realm of physics apart. But as is now evident, we cannot allow an arbitrary amount of information to be carried by hidden variables. In order to make a fair distinction, the amount of classical communication used while attempting to simulate a phenomenon should be appropriately chosen.

We will, therefore, restrict the information content of the classical system that is measured at different instances in time to the number of bits accessible from a qubit which is 1 bit (since we restrict this discussion to quantum states that are qubits).

Chapter 3

Finite Shared Randomness

3.1 GHZ Paradox

3.1.1 GHZ State

An entangled state of the form[6]

$$|\psi\rangle = \frac{1}{\sqrt{2}}(|0\rangle^{\otimes n} + |1\rangle^{\otimes n}) \tag{3.1}$$

is referred to as the GHZ state.

3.1.2 The Paradox

Consider, for simplicity $n = 3$. From (1.13) to (1.18), within a local hidden variables framework, we have

$$\begin{aligned} r_1(x,\lambda)r_2(y,\lambda)r_3(y,\lambda) &= -1, & (3.2)\\ r_1(y,\lambda)r_2(x,\lambda)r_3(y,\lambda) &= -1, & (3.3)\\ r_1(y,\lambda)r_2(y,\lambda)r_3(x,\lambda) &= -1, & (3.4) \end{aligned}$$

where x and y stand for measurement along the x and y directions respectively.

Taking products of both sides of the above equations, we obtain

$$r_1(x,\lambda)r_2(x,\lambda)r_3(x,\lambda) = -1. \tag{3.5}$$

However, from (1.13) to (1.18) it is required that

$$r_1(x,\lambda)r_2(x,\lambda)r_3(x,\lambda) = 1 \tag{3.6}$$

This leads to a contradiction which can be used to disqualify the local hidden variables theory.

3.1.3 GHZ Correlations

Consider $\mathbf{s} \in \{\mathbb{I}, x, y, z\}^{\otimes n}$ where $\mathbb{I}$ represents no measurement made and x, y and z represent measurements along x, y and z respectively. Let

$$n_s = \sum_{t=1}^{n} \delta_s^{s_t} \tag{3.7}$$

represent the number of measurements of kind s, where $n = \sum_s n_s$

From (3.1) and (1.13) to (1.18) we can derive the following correlations.

- If $n_{\mathbb{I}} + n_z > 0$ and $n_x + n_y > 0$, $E(\mathbf{s}) = 0$.
- If $n_x + n_y = 0$, $E(\mathbf{s}) = 1 - n_z \bmod 2$.
- If $n_{\mathbb{I}} + n_z = 0$, $E = Re(\iota^{n_y})$.

This is due to the following:

- Measurement operators collapse the state into their eigenbasis. In this case measurements are carried out individually on entangled systems. Each system will collapse according to the measurement carried out on it. In order for the correlations in measurement outcomes to yield a non-zero value, the collapsed state should be (3.1).
- Measurements along x and y swaps $|0\rangle$ and $|1\rangle$ whereas measurements along z do not. Any combination of $\{x, y\}$ and $\{\mathbb{I}, z\}$ would therefore never result in (3.1) as the state after measurement.

3.2 Temporal GHZ Correlations

The work by Markiewicz et. al.[7] provides a method of construction that can map a certain class of spatial correlations to temporal correlations. The GHZ correlations belong to this class. Such a map uses generalised Positive Operator Valued Measures.

The existence of such a map justifies investigating whether we can obtain GHZ correlations using deterministic temporal protocols.

3.2.1 Method

The information content of $\lambda \in \{0, 1\}$ is restricted to 1 bit. The following protocol is checked.

1. A function is chosen which describes, for each t, $r_t(s_t, \lambda_t) \in \{-1, 1\}$ and $\lambda_t(s_t, \lambda_{t-1})$. This choice of function is parametrised by μ.
2. For each value of $(\mathbf{s}, \lambda_1, \mu)$, $\mathbf{r}(\mathbf{s}, \lambda_1, \mu)$ was obtained.
3. For each μ, correlations were checked by averaging over λ_1 uniformly for all $\mathbf{s}$ for all possible combinations of timesteps. Lower order correlations are represented by replacing the measurements not chosen by $\mathbb{I}$.

3.3 Results

3.3.1 $M = 2$

Consider the allowed measurements to be along x and y.

$n = 2$

Several functions were obtained for which all correlations in agreement with the GHZ correlations.

$n = 3$

The total number of possible functions is given by 2^{2nM} where M is the number of measurement settings.

No function was obtained for which all correlations were in agreement with the GHZ correlations.

However, several functions were obtained for which all the non-zero correlations matched. This is of interest since μ, that parametrises the functions, can be considered as shared randomness. An appropriate measure, $p(\mu)$, can always be found such that an average over them reproduces the zero correlations as well. The non-zero ones will always match since only those functions that satisfy them will be sampled. For the others, $p(\mu) = 0$.

Here, the number of possible functions is 2^{12}. Therefore, there are 12 bits of shared randomness, which is finite unlike the usually investigated problem of infinite shared randomness.

3.3.2 $M = 3$

Consider measurement settings along all 3 directions, x, y and z.

Even for $n = 2$, there was no single protocol that reproduced just the non-zero correlations.

Here, there are 2^{12} possible functions and hence 12 bits of shared randomness.

Therefore, no classical protocol with 12 bits of shared randomness can reproduce GHZ correlations for 3 measurement settings even when 1 bit is communicated.

Even if the shared randomness was to manifest itself in any other way, any evenly distributed manifestation (which does not result in groupings such that multiple functions yield the same outcome) will be equivalent to expressing shared randomness in this manner.

Chapter 4

Bell's Inequality with Communication

Let us review the expression for correlations in the context of temporal processes.

$$E(\mathbf{s}) = \sum_{\lambda_1,\mu} p(\lambda_1)p(\mu) \sum_{\mathbf{r}} \prod_t r_t p_t(r_t|s_t;\lambda_t;\mu). \tag{4.1}$$

4.1 Convex Geometry

Definition. *A* ***convex combination*** $\sum_i c_i p_i$ *of points* $\{p_i\}$ *in an affine space is a linear combination with the restriction* $c_i \in [0,1]$ *and* $\sum_i c_i = 1$.

Theorem. *Any convex combination of* $\{p_i\}$ *will lie on or inside the convex polytope formed with* $\{p_i\}$ *as its vertices.*

4.2 Joint Probabilities

Due to the form of the correlation, it is useful to define joint probabilities

$$p(\mathbf{r}|\mathbf{s};\mu) = \prod_{t=1}^{n} p_t(r_t|s_t;\lambda_t;\mu), \tag{4.2}$$

where n is the number of timesteps since $\lambda_t = \lambda_t(\lambda_{t-1}, s_t)$. Here, we fix the value of λ_1 to a constant. Hence, for each μ we have entirely deterministic protocols and $p(\mathbf{r}|\mathbf{s};\mu) \in \{0,1\}$.

Consider the affine space $[0,1]^{\otimes n}$. For each μ, we can generate a point in the space to represent joint probabilities obtained for each $(\mathbf{r}, \mathbf{s})$ where $n = |\{(\mathbf{r}, \mathbf{s})\}|$[8].

The value of n can be reduced by considering the following:

- Probability is normalised. One value of $\mathbf{r}$ can be eliminated.
- Outcomes of the past are independent of those of the future which can be used to eliminate joint probabilities in favour of marginal probability distributions,

$$p(\mathbf{r}|\mathbf{s}) = \sum_{i=t+1}^{n} \sum_{r_i=0}^{K-1} p(\mathbf{r}, \mathbf{r}'|\mathbf{s}, \mathbf{s}'), \tag{4.3}$$

  where $\mathbf{r}$ is the set of outcomes up to t, $\mathbf{s}$ is the set of settings up to t, $\mathbf{r}'$ is the set of outcomes after t and $\mathbf{s}'$ is the set of outcomes after t. Here, in the marginal probability distributions, dependence on settings of future timesteps is dropped.

It will turn out later that this elimination is not just a mathematical advantage that will take the form of a computational advantage but is a physical requirement which if not accounted for, will make the problem unsolvable.

For any given probability distribution $p(\mu)$, the set of correlations obtained would lie on or inside the convex polytope generated by these points.

4.3 The Protocol

Consider the following deterministic protocol.

1. At timestep t, a measurement setting $s_t \in \mathbb{Z}_{M_t}$ is chosen where M_t is the number of measurement settings that are available at t.
2. λ_t is received as communication from the previous timestep.
3. The outcome $r_t(s_t, \lambda_t, \mu) \in \mathbb{Z}_{K_t}$ is provided where K_t is the number of possible outcomes.
4. $\lambda_t(s_t, \lambda_{t-1}, \mu)$ is sent as communication to the next timestep.

The protocol mentioned above is very general. Based on physical requirements, we add the following constraints.

- $K_t = K$ keeping in mind that the objective is to simulate quantum phenomenon and the number of states a system can attain is constant.
- Amount of information carried by λ_t, $l = \lceil \log_2 K \rceil$. This also ensures that $K < 2^l \times M$. This condition avoids obtaining several inequalities that are equivalent to one another.

Notice that it is not the functional form of λ_t that is of consequence but the outcome that it leads to. Multiple functions may lead to the same outcome. Such groupings in μ may be identified and used to reduce the number of points which would overlap at a vertex. This is advantageous for longer protocols that are computationally heavy.

4.4 Inequalities of Interest

Once the vertices of the convex polytope are generated, the facets of this polytope can be obtained. The facets restrict the probabilities to a half-space and the equation of the plane which can be used to obtain an inequality that will be obeyed by all vertices.

Some of these facets correspond to trivial inequalities which are the non-negativity conditions or those that compare marginals to the joint probabilities they are composed of. It is a mathematical requirement that these inequalities never be violated and hence they are not of interest.

There are other non-trivial inequalities of the form

$$\sum_i c_i p_i \leq \sum_i c_i, \tag{4.4}$$

when $c_i \geq 0$ which arise simply from the condition that $p_i \leq 1$ which can also never be violated.

Of interest are the inequalities of the kind

$$\sum_i c_i p_i \leq d, \tag{4.5}$$

where

$$d < \sum_i c_i \tag{4.6}$$

for which, mathematically, probability distributions may be found that violate them. That leaves scope for violation by quantum phenomenon.

4.5 Methods of Computation

Numerical methods were used to obtain these inequalities. It is computationally very heavy if all inequalities were to be obtained. This is why, in every instance of the protocol, only one inequality is obtained. Randomness is included wherever possible to allow for a different inequality to be obtained for each instance.

The following protocol was used.

1. The deterministic protocol is followed for every possible sequence of all possible functions operating at each step to provide an outcome and operate on the state. Based on the outcomes vertices are generated.
2. Vertices picked at random are added one after the other to a set ensuring that they define a hyperplane of dimension equal to number of vertices in the set.
3. Once n vertices are picked, they are used to define an $(n-1)$D plane.
4. A point from the plane is chosen at random and is replaced by another point.
5. 4 is continued till all the vertices are on one side of the plane.
6. The plane obtained is a facet of the polytope and thus the inequality is obtained.

Avoiding Trivial Inequalities

In order to ensure that we do not obtain trivial inequalities, the following constraints are added to the protocol.

At 2, when vertices are chosen to form the set that would eventually define an $(n-1)$D plane, they are chosen such that this plane does not contain the point $0^{\otimes n}$ in order to avoid trivial inequalities. A vertex, p is picked to occupy the ith point in the set only when $p_i = 1$ while forming the initial plane. Also, it is ensured mathematically that such a set of vertices will always exist.

Theorem. *In order for a set of i vertices to define a iD plane that does not contain $0^{\otimes n}$, the otherwise only sufficient condition that rank of the matrix formed by the vertices as rows be equal to i is necessary.*

Due to the condition on vertices, the plane formed will never contain $0^{\otimes n}$ and hence, the condition on rank may be used.

At 4 it is useful to treat the point $0^{\otimes n}$ as the *origin* inspite of the space being affine since the point is privileged as seen above. The vertex chosen at random should have a length greater than the distance of the plane from the origin. This condition ensures that the protocol does not take arbitrarily long to run by providing a direction of convergence to a facet.

At 4 it is also ensured that the vertex p chosen at random can replace the vertex at i only if $p_i = 1$.

There are no checks to eliminate inequalities of the kind in (4.4). Often, those may be the only inequalities apart from trivial inequalities.

4.6 Results

4.6.1 CHSH Inequality

The protocol was tested for the following parameters:

- $n = 2$.
- $K = 2$.
- $M_1 = M_2 = 2$.
- No communication was allowed. $l = 0$.

This is equivalent to spacelike separated measurements made with infinite shared randomness.

Of the inequalities obtained, the following is of significance:

$$2p(0|1) - p(00|00) + p(10|00) - p(00|10) + p(10|10) + p(00|01) \\ -p(00|11) + p(10|11) \leq 2. \qquad (4.7)$$

From (1.20) this is the CHSH inequality[9].

4.6.2 Trivial Problem

The protocol was run the following parameters:

- $n \in \{2, 3, 4\}$.
- $K = 2$.
- $M_t = 2$ for all t.
- $l = 1$ since $K = 2$.

Only trivial inequalities or those of the kind in (4.4) were obtained in all cases.

The reason behind this is that the amount of information sent from one step to the next is sufficient to exactly specify what measurement setting was selected and the outcome at each step can be determined accordingly.

It is expected that for any number of timesteps, only trivial inequalities or inequalities of the kind in (4.4) will be obtained.

4.6.3 Simplest Non-trivial Problem

The protocol was run for the following parameters[10]:

- $n = 2$.
- $K = 2$.
- $M_1 = 3$ and $M_2 = 2$.

- $l = 1$ since $K = 2$.

This is the simplest non-trivial problem.

Of the inequalities obtained, the following are of interest:

$$\begin{aligned} -p(0|0) + p(00|00) - p(00|10) - p(10|10) - p(00|20) + p(10|20) \\ +p(00|01) - p(00|11) - p(10|11) + p(00|21) - p(10|21) \leq 1, \end{aligned} \tag{4.8}$$

$$\begin{aligned} -3p(0|0) + 2p(00|00) - p(10|00) - 2p(00|10) - p(10|10) + p(10|20) \\ +2p(00|01) - p(10|01) + p(00|11) - p(00|21) - 2p(10|21) \leq 1. \end{aligned} \tag{4.9}$$

Chapter 5

Bounded Accuracy when No Measurement is Made

We introduce a problem[11] in time and investigate the accuracy with which classical and quantum computers can solve it. This problem, unlike the others, does not involve measurements made at every step but just requires the computer to provide an output upon termination. Every step in the protocol, therefore, is an operation and the value of the processing unit of the computer at the end of the protocol is taken to be the output.

5.1 The Problem

$(\mathbf{s^1}, \mathbf{s^2})$ is provided progressively at each timestep t as inputs where $s_t^1, s_t^2 \in \{0, 1\}$ to classical and quantum processors with 1-bit capacity, $\lambda_t \in \{-1, 1\}$. The process may be declared terminated at any instance n. Then, the output is the value λ_n. It is required to be of the form

$$T_n = \cos\left(\frac{\pi}{2}\sum_{j=1}^{n} s_j^1\right)(-1)^{\sum_{i=1}^{n} s_i^2}, \tag{5.1}$$

under the promise that

$$p(\mathbf{s^1}) = 2^{-n+1}\left|\cos\left(\frac{\pi}{2}\sum_{j=1}^{n} s_j^1\right)\right|, \tag{5.2}$$

where $p(\mathbf{s^1})$ is the probability of obtaining the bit sequence $\mathbf{s^1}$. It essentially requires that $\sum_{i=1}^{n} s_i^1 \bmod 2 = 0$. We ensure a uniform probability distribution over all values of $\mathbf{s^2}$.

5.2 Bound on Accuracy for Classical Computers

The quantity of interest is

$$F = \langle \lambda_n T_n \rangle, \tag{5.3}$$

because

$$\lambda_n T_n = \begin{cases} +1, \text{ if the output is correct} \\ -1, \text{ otherwise} \end{cases} . \tag{5.4}$$

We have

$$F = \sum_{(\mathbf{s^1},\mathbf{s^2}) \in \{0,1\}^{\otimes 2n}} 2^{-n} p(\mathbf{s^1}) \lambda(\mathbf{s^1}, \mathbf{s^2}) T_n. \tag{5.5}$$

Given the limitations on processor capacity, we may only allow a Markov chain such that

$$\begin{aligned} \lambda_t &= \lambda_t(\lambda_{t-1}; s_t^1, s_t^2) && (5.6) \\ &= D_{s_t^1, s_t^2} + C_{s_t^1, s_t^2} \lambda_{t-1}. && (5.7) \end{aligned}$$

Since $\lambda_t = \pm 1$,

$$|C_{s_t^1, s_t^2}| + |D_{s_t^1, s_t^2}| = 1. \tag{5.8}$$

We constraint the form of $C_{s_t^1, s_t^2}$ such that it can be split further as

$$C_{s_t^1, s_t^2} = e_t(s_t^1) + \frac{1}{2}(-1)^{s_t^2} c_t(s_t^1), \tag{5.9}$$

where $s_t^2 \in \{0, 1\}$ translates to $(-1)^{s_t^2} \in \{-1, 1\}$ requiring

$$c_t(s_t^1) = \pm 1. \tag{5.10}$$

Then for $t > 1$

$$\sum_{(s_{t-1}^2, s_t^2) \in \{0,1\}^{\otimes 2}} (-1)^{s_{t-1}^2 + s_t^2} \lambda_t$$

$$= \sum_{(s_{t-1}^1, s_t^2) \in \{0,1\}^{\otimes 2}} (-1)^{s_{t-1}^2 + s_t^2} C_{s_t^1, s_t^2} \lambda_{t-1} \tag{5.11}$$

$$= \sum_{s_{t-1}^2 \in \{0,1\}} (-1)^{s_{t-1}^2} c_t(s_t^1) \lambda_{t-1}. \tag{5.12}$$

The non-vanishing terms are due to (5.6).

By induction,

$$\lambda_t = (-1)^{s_t^2} c_t(s_t^1) \lambda_{t-1}. \tag{5.13}$$

Without loss of generality, $\lambda_1 = 1$. This results in

$$F = \sum_{\mathbf{s^1} \in \{0,1\}^{\otimes n}} f(\mathbf{s^1}) \prod_{i=1}^{n} c_i(s_i^1), \tag{5.14}$$

which is bounded as

$$F \leq 2^{-N+1}, \tag{5.15}$$

where

$$N = \begin{cases} \frac{n+1}{2} & \text{, if } n \bmod 2 = 1 \\ \frac{n}{2} & \text{, otherwise} \end{cases}. \tag{5.16}$$

Here, correlations in values of c can be considered analogous to correlations in measurement outcomes in Bell's inequality.

5.3 Violation of the Bound by Quantum Computers

For a quantum computer we may initialise the qubit to the state

$$|\psi_0\rangle = 2^{-1/2}(|0\rangle + |1\rangle). \tag{5.17}$$

If at each t the unitary phase-shift transformation $|0\rangle\langle 0| + \exp(\iota(\pi/2)(2s_t^2 + s_t^1))|1\rangle\langle 1|$ is applied on the qubit, we obtain

$$|\psi_t\rangle = 2^{-1/2} \left(|0\rangle + \exp\left(\iota \frac{\pi}{2} \sum_{i=1}^{t} (2s_i^2 + s_i^1) \right) |1\rangle \right). \tag{5.18}$$

Upon termination of the process, a measurement made in the basis $\{2^{-1/2}(|0\rangle + |1\rangle), 2^{-1/2}(|0\rangle - |1\rangle)\}$ would provide the exact result.

Hence, a quantum computer allows

$$F = 1, \tag{5.19}$$

exceeding the bound obtained for classical computers.

Here, the property of a qubit that allows it to possess more information than what is accessible is exploited. Upon measurement, it may only be possible to retrieve 1 bit but the promise ensures that the qubit will be in either of the states that form the basis of measurement.

5.4 Allowing Measurements at Every Timestep

In order to form a connection with the temporal inequalities discussed previously, we investigate the effect of allowing measurements to be made at every timestep.

We define a problem where

$$T_n = \left(\sum_{i=1}^{n} s_i \right) \bmod 4, \tag{5.20}$$

where $s_t = 2y_t + x_t$ is the 2-bit input provided at each t where $x_t, y_t \in \{0, 1\}$. Here, $\lambda \in \{0, 1\}$.

The final result is obtained as a function of the measurement output r_t at each t.

Similar to the original problem, the promise holds as in (5.2) and the output may be demanded at any instance of time.

This problem is equivalent to the original problem because of the following:

- Both problems input 2 bits at each timestep.
- Both problems use a processor with 1-bit capacity.

- The form of the expression for both problems require the promise to restrict the output to 1 bit.

We choose the following protocol:

$$\lambda_t = \lambda_{t-1} \oplus r_t, \tag{5.21}$$

where if we let $A_0 = 0$ without loss of generality it ensures that the processor stores the parity of $\sum_{i=1}^{t} s_t$.

$$r_t = y_t \oplus x_t \oplus \lambda_{t-1} \cdot x_t. \tag{5.22}$$

Then, the final result may be provided as

$$\begin{aligned}
&2\left(\left(\sum_{i=1}^{n} r_i\right) \bmod 2\right) \\
&= \left(2\left(\left(\sum_{i=1}^{n}(y_i + x_i + \lambda_{i-1} \cdot x_i)\right) \bmod 2\right)\right) \bmod 4 && (5.23)\\
&= \left(2\sum_{i=1}^{n}(y_i + x_i + \lambda_{i-1} \cdot x_i)\right) \bmod 4 && (5.24)\\
&= \left(\sum_{i=1}^{n} s_i\right) \bmod 4 && (5.25)\\
&= T_n, && (5.26)
\end{aligned}$$

since

$$\left(\sum_{i=1}^{n}(x_i + 2\lambda_{i-1} \cdot x_i)\right) \bmod 4 = 0$$

due to the promise.

Therefore, if we allow for measurements to be made at every stage, even a classical computer can solve the problem exactly as shown.

Chapter 6

Conclusion

6.1 Summary

The main results obtained in the direction of understanding the limits of classical mechanics in explaining temporal phenomena can be summarised as follows:

- Understanding the gains provided by communication
- Demonstrating that GHZ correlations can be reproduced by classical protocols
- Proposing a method to obtain inequalities similar to Bell's inequalities that may be checked for violation experimentally
- Deriving a bound on accuracy that a classical protocol can achieve while solving a given problem

6.2 Applications

An understanding of temporal phenomena unique to quantum mechanics can have several applications such as:

- It deepens understanding of the extent to which analogies between space and time can be formed and the effect measurement-making has on the arrow of time.

- It can guide investigations into the benefits of using quantum computers.
- It can open up the possibilities to carry out sequential computing that cannot be handled by classical computers.

Chapter 7

Future Inquiries

7.1 Ergodicity

It can be investigated whether it is possible to represent temporal evolution of states by maps. If possible, the maps can be checked for ergodicity. Ergodic maps would require the expected temporal correlations to be identical to spatial correlations obtained for the same distribution of states which can be obtained more easily.

7.2 Bounded Error

Any inequality that is obtained and can be shown to be violated theoretically in the temporal scenario is expected to involve a large number of measurements making it experimentally and probably even numerically impractical to verify or even to exploit in applications.

Instead, problems may be proposed that are to be solved by classical sequential computing and quantum sequential computing and in both cases

$$p(n) < \epsilon \tag{7.1}$$

where $p(n)$ is the probability of an error in the solution given complexity n and ϵ is the bound on the error. These bounds can be compared instead of comparing correlations exactly. Differences in these bounds may show up even for few measurements.

7.3 Amount of Finite Shared Randomness Required to Obtain GHZ Correlations

In 3 the amount of shared randomness among measuring parties was finite.

The amount of shared randomness can be increased by μ' such that

$$f_\mu = f_\mu(\mathbf{r}|\mathbf{s}; \lambda_1; \mu'), \tag{7.2}$$

where f_μ is the function that generates the outcome that is parametrised by μ. Here, μ' manifests itself as another parameter that affects measurement outcome.

It can be investigated to find the amount of this shared randomness, in addition to possessing information about the choice of function at each step, required to be able to simulate GHZ correlations.

The results can be verified experimentally.

7.4 Obtaining Inequalities for a Sequence of Observers

The method described in 4 can be used to obtain inequalities for sequences of any number of measurements. However, it is a very computationally demanding task.

7.5 Violation of Bounds on Classical Processes by Quantum Processes

The method described in 4 obtains inequalities. These inequalities place bounds on what can be achieved classically. Experiments may be carried out to investigate which of these bounds can be violated by quantum processes.

7.6 Reducing Number of Measurements Required

The protocol[4] that makes sequential general measurements on quantum systems, correlations on which cannot be obtained classically had certain drawbacks as listed previously in 2.2.2.

The conditions on classical protocols can be made stronger by requiring that local probability distributions are reproduced. In that case classical mechanics may be disqualified even for a shorter sequence of measurements. It would be more feasible to simulate such measurements numerically and maybe even make them experimentally.

Bibliography

[1] J. Bell, Physics 1, 195 (1964).

[2] A. Einstein, B. Podolsky and N. Rosen, Phys. Rev. 47, 777 (1935).

[3] A. J. Leggett and A. Garg, Phys. Rev. Lett. 54, 857 (1985).

[4] S. Brierley, A. Kosowski, M. Markiewicz, T. Paterek, and A. Przysiężna, PRL 115, 120404 (2015).

[5] B. F. Toner and D. Bacon, Phys. Rev. Lett. 91, 187904 (2003).

[6] D. Greenberger, M. A. Horne, A. Zeilinger, arXiv.0712.0921 (2007).

[7] M. Markiewicz, A. Przysiężna, S. Brierley, and T. Paterek, Phys. Rev. A 89, 062319 (2014).

[8] D. Bacon and B. F. Toner, Phys. Rev. Lett. 90, 157904 (2003).

[9] J. Clauser, M. Horne, A. Shimony, and R. Holt, Phys. Rev. Lett. 23, 880 (1969).

[10] K. Maxwell and E. Chitambar, Phys. Rev. A 89, 042108 (2014).

[11] M. Żukowski, Front. Phys. 9, 629 (2014).